1. Prove that convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$. Is the converse true?

NTS: $\{S_n\}$ is cgt $\Rightarrow$ $\{|S_n|\}$ is cgt

We use ε- defintion

Let $\varepsilon > 0$.

~~Since $\{S_n\}$ is cgt~~

Suppose that $\{S_n\}$ is cgt to S.

Then $\exists\, N_0$ s.t. $\cancel{|S_n - S|}$ $n \geq N_0 \Rightarrow |S_n - S| < \varepsilon$

Suppose that $n \geq N_0$, Then

$$\big||S_n| - |S|\big| \leq |S_n - S| < \varepsilon \quad \text{(Reverse triangle inequlity)}$$

Thus, $\{|S_n|\}$ is cgt to $|S|$.

But The converse in false.

Converse: If $\{|S_n|\}$ is cgt then $\{S_n\}$ is cgt.

Counter example: Let $\{S_n\}=\{(-1)^n\}$. Then,

$|S_n| = |(-1)^n| = 1$ for all $n \in \mathbb{N}$

Then $|S_n| \to 1$ as $n \to \infty$.

i.e: S_n is cgt to 1.

But $\{S_n\}$ is not cgt. We prove that by contradiction

Suppose that $\lim_{n\to\infty} (-1)^n = L$

Then $\exists N_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}, n \geq N_0 \Rightarrow |(-1)^n - L| < 1$

So, $|(-1)^n - L| < 1$

Furthur $|(-1)^{n+1} - L| < 1$

We know that either $((-1)^n = 1$ and $(-1)^{n+1} = -1)$ or $((-1)^n = -1$ and $(-1)^{n+1} = 1)$. Therefore,

$|1 - L| < 1$ and $|-1 - L| < 1$

$-1 < (1 - L) < 1$ and $-1 < (1 + L) < 1$

Thus, $-L < 0$ and $L < 0$

$L > 0$

This contradiction

2. Calculate $\lim\limits_{n\to\infty} (\sqrt{n^2+n} - n)$.

Calculate $\lim\limits_{n\to\infty} (\sqrt{n^2+n} - n)$

Observe

$$\sqrt{n^2+n} - n = \frac{(\sqrt{n^2+n}+n)}{(\sqrt{n^2+n}+n)} \times (\sqrt{n^2+n} - n)$$

$$= \frac{(n^2+n) - n^2}{(\sqrt{n^2+n}) + n}$$

$$= \frac{n}{(\sqrt{n^2+n}) + n}$$

$$= \frac{1}{\left(\sqrt{1+\frac{1}{n}}\right) + 1}$$

Then

$$\lim_{n\to\infty} (\sqrt{n^2+n} - n) = \lim_{n\to\infty} \frac{1}{\left(\sqrt{1+\frac{1}{n}}\right) + 1} = \frac{1}{2}$$

3. If $s_1 = \sqrt{2}$, and

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} \qquad (n = 1, 2, 3, \ldots),$$

prove that $\{s_n\}$ converges, and that $s_n < 2$ for $n = 1, 2, 3, \ldots$.

the question
in this column.

(3) Suppose that $S_1 = \sqrt{2}$ and

$$S_{n+1} = \sqrt{2 + \sqrt{S_n}} \quad \text{for } n = 1, 2, 3, \ldots$$

~~We are going to proo~~ use thm 3.14.
First we are going to show that, $S_n < 2$, $n = 1, 2, \ldots$
We are going to use mathematical induction.
$S_1 = \sqrt{2} < 2$ (Base is trivial)
Assume that $S_p < 2$ ~~f~~ $p \in \mathbb{N}$.
Then $p+1$ case

$$S_{p+1} = \sqrt{2 + \sqrt{S_p}} < \sqrt{2 + \sqrt{2}} < 2$$

Therefore, by mathematical induction,
$S_n < 2$ for all $n \in \mathbb{N}$.

Since we already show that $\{S_n\}$ is ~~o~~ bdd
It is ~~e~~ easy to use thm 3.14 So we have
to show that $\{S_n\}$ is montone Again we
use the mathematical induction.

$S_1 = \sqrt{2}$

$S_2 = \sqrt{2+\sqrt{S_1}} = \sqrt{2+\sqrt{\sqrt{2}}}$

$= (2 + 2^{1/4})^{1/2} > \sqrt{2} = S_1$

So, base case is Truse.

Now assume that

$S_{p+1} > S_p \quad , p \in \mathbb{N}$ — ⊛

2+

Now we need to show that $S_{p+2} > S_{p+1}$

$$S_{p+2} = \sqrt{2+\sqrt{S_{p+1}}} > \sqrt{2+\sqrt{S_p}} \text{ (by ⊛)}$$

$$= S_{p+1}$$

Therefore, by mathematical induction,

$S_{n+1} > S_n$ for all $n = 1, 2, 3 \ldots$

Theefore, $\{S_n\}$ is monotonically increasing and bound above. Therefore by Thm 3.14 $\{S_n\}$ is convergent

4. Find the upper and lower limits of the sequence $\{s_n\}$ defined by

$$s_1 = 0; \qquad s_{2m} = \frac{s_{2m-1}}{2}; \qquad s_{2m+1} = \frac{1}{2} + s_{2m}.$$

(4) $S_1 = 0$

$S_{2m} = \frac{S(2m-1)}{2}$

$S_{2m+1} = \frac{1}{2} + S_{2m}$

$S_1 = 0$
$S_2 = 0$
$S_3 = \frac{1}{2}$
$S_4 = 1/4$
$S_5 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

Claim: $S_{2m} = \frac{1}{2} - \frac{1}{2^m}$ for all $m = 1, 2, 3$

We are going to use mathematical induction

$S_2 = \frac{1}{2} - \frac{1}{2^1} = 0$. Thus, ~~m~~ base case is true.

Now suppose that $S_{2p} = \left(\frac{1}{2} - \frac{1}{2^p}\right)$

Now $p+1$ case,

$$S_{2^{(p+1)}} = \frac{S_{2^{p+1}}}{2}$$

$$= \frac{1}{2}\left(\frac{1}{2} + S_{2^p}\right)$$

$$= \frac{1}{4} + \frac{1}{2} S_{2^p}$$

$$= \frac{1}{4} + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2^p}\right) \quad \text{(by induction hypothesis)}$$

$$= \frac{1}{4} + \frac{1}{4} - \frac{1}{2^{p+1}}$$

$$= \frac{1}{2} - \frac{1}{2^{p+1}}$$

Then By mathematical induction,

$S_{2m} = \frac{1}{2} - \frac{1}{2^m}$ for $m \in \mathbb{N}$. Then

$$S_{2m+1} = \frac{1}{2} + S_m = \frac{1}{2} + \frac{1}{2} - \frac{1}{2^m} = 1 - \frac{1}{2^m} \text{ for } m \in \mathbb{N}.$$

~~Then~~ $\cancel{\lim_{m \to \infty} S_{2m} = \frac{1}{2} - \frac{1}{2^m} = \frac{1}{2}}$

$$\cancel{\lim_{m \to \infty} S_{2m+1} = 1 - \frac{1}{2^m}}$$

Then,
$$\lim_{m \to \infty} S_{2m} = \lim_{n \to \infty} \left(\frac{1}{2} - \frac{1}{2^m}\right) = \frac{1}{2}$$

$$\lim_{m \to \infty} S_{2m+1} = \lim_{n \to \infty} \left(1 - \frac{1}{2^m}\right) = 1$$

Hence,

$$\limsup_{n \to \infty} (S_n) = 1 \quad \text{and}$$

$$\liminf_{n \to \infty} (S_n) = \frac{1}{2}$$

Let $\{a_n\}$ and $\{b_n\}$ be two real sequnce.

~~Case~~

$\boxed{\text{Avoid cases}}$ $*$ If $\limsup\limits_{n\to\infty} a_n = \infty$ and $\limsup\limits_{n\to\infty} b_n = -\infty$

$*$ If $\limsup\limits_{n\to\infty} a_n = -\infty$ and $\limsup\limits_{n\to\infty} b_n = \infty$

$\boxed{\text{Case-I}}$ - If $\limsup\limits_{n\to\infty} a_n = +\infty$ or $\limsup\limits_{n\to\infty} b_n = +\infty$

~~- If $\limsup\limits_{n\to\infty} a_n = -\infty$ and $\limsup\limits_{n\to\infty} b_n = -\infty$~~

Then result is obviou.

$\boxed{\text{Case-II}}$ - If $\limsup\limits_{n\to\infty} a_n = -\infty$ and b_n is bounded above. (i.e. $\exists M_0$, $b_n \leq M_0$ for all $n \in \mathbb{N}$.

$$\left(\limsup_{n\to\infty} a_n = -\infty\right) \Rightarrow \left(\begin{array}{l}\forall M \in \mathbb{R}\ \exists N \in \mathbb{N} \text{ s.t.} \\ n \geq N \Rightarrow a_n \leq M\end{array}\right)$$

$(a_n + b_n) \leq a_n + M_0 \leq M + M_0 =$

~~Note~~ Then, $\forall M \in \mathbb{R}$, $\exists N \in \mathbb{N}$ s.t.

$n \geq N \Rightarrow (a_n + b_n) \leq M + M_0$

~~$\lim\limits_{n\to\infty} S_n = -\infty$~~

Therefore $\lim\limits_{n\to\infty} (a_n + b_n) = -\infty$.

So given result is true for this case.

5. For any two real sequences $\{a_n\}$, $\{b_n\}$, prove that

$$\limsup_{n\to\infty}(a_n+b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n,$$

provided the sum on the right is not of the form $\infty - \infty$.

(5) Let $\{a_n\}$, $\{b_n\}$ be a two real sequen[ces]

We ignore the case right hand side, form $\infty-\infty$

Case-I

If $\limsup_{n\to\infty} a_n = +\infty$ and $\limsup_{n\to\infty} b_n = +\infty$

The inequlity trivially holds.

Case-II one of $\limsup_{n\to\infty} a_n$, $\limsup_{n\to\infty} b_n$ is $-\infty$ and other one is $<\infty$.

Then $\{a_n+b_n\}$ is tends to $-\infty$

Then inequlity holds.

Case-III If $\limsup_{n\to\infty} a_n = -\infty$ and $\limsup_{n\to\infty} b_n = -\infty$

Then also inequlity holds.

Case-IV If $\lim \cancel{\sup} \sup a_n < \infty$ and $\limsup b_n < \infty$

This is the only ~~ent~~ non-trivial case.

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Let E be the set of subsequnetial limits of $\{a_n + b_n\} = \{c_n\}$

Suppose that $l \in E$.

Then there exist a subsequnce $\{c_{n_k}\} = \{a_{n_k} + b_{n_k}\}$ of sequnce of $\{c_n\} = \{a_n + b_n\}$.

such that

$$\lim_{k \to \infty} c_{n_k} = \lim_{k \to \infty} (a_{n_k} + b_{n_k}) = l.$$

Now we consider the sequnce $\{a_{n_k}\}$

Since $\{a_n\}$ is bounded. The subsequnce $\{a_{n_k}\}$ is also bounded. Thus there exist subsequn $\{a_{n_{k_l}}\}$ of the sequnce $\{a_{n_k}\}$ such that is convergent (Thm 3.6)

Sine $\{c_{n_k}\}$ is cgt. the subseqnce $\{c_{n_{kp}}\}$ is cgt.

Thus $\{b_{n_{kp}}\}$ is also cgt. Because

$$b_{n_{kp}} = c_{n_{kp}} - a_{n_{kp}}$$

$$b_{n_{kp}} = (a_{n_{kp}} + b_{n_{kp}}) - a_{n_{kp}}$$

Hence, $l = \lim_{k \to \infty} c_{n_k}$

$$l = \lim_{k \to \infty} (a_{n_k} + b_{n_k})$$

$$= \lim_{p \to \infty} (a_{n_{k_p}} + b_{n_{k_p}})$$

$$= \lim_{p \to \infty} a_{n_{k_p}} + \lim_{p \to \infty} b_{n_{k_p}}$$ (Since both $\{a_{n_{k_p}}\}$, $\{b_{n_{k_p}}\}$ is cgt)

$$\leq \limsup_{n \to \infty} a_n + \limsup_{p \to \infty} b_n$$

Since l is an arbitray element from E.

$\limsup_{n \to \infty} a_n$ and $\limsup_{n \to \infty} b_n$ are upper bound of the set E. By the def^n sup

$$\limsup_{n \to \infty} (a_n + b_n) \leq \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n$$

6. Investigate the behavior (convergence or divergence) of Σa_n if

(*a*) $a_n = \sqrt{n+1} - \sqrt{n}$;

(*b*) $a_n = \dfrac{\sqrt{n+1} - \sqrt{n}}{n}$;

(*c*) $a_n = (\sqrt[n]{n} - 1)^n$;

(*d*) $a_n = \dfrac{1}{1 + z^n}$, for complex values of z.

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$$a_n = \sqrt{n+1} - \sqrt{n}$$

a)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n})$$

$$= \lim_{n\to\infty} \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}+1-\cancel{n}}{\sqrt{n+1}+\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

$$= 0$$

So I thought we can use thm 3.23 but by the result, we cannot use it.
By this result we can not conulder the convergence or divergence.
Now we consider partial sums,

$$S_n = \sum_{k=1}^{n} \sqrt{k+1} - \sqrt{k}$$

$$= (\cancel{\sqrt{2}} - \sqrt{1}) + (\sqrt{3} - \cancel{\sqrt{2}}) + \cdots + (\sqrt{n+1} - \cancel{\sqrt{n}})$$

$$= \sqrt{n+1} - 1$$

Claim: S_n is not-bounded above.

Let $M \in \mathbb{R}$
then we can find $n_0 \in \mathbb{Z}^+$ such that,

$$n_0 > (M+1)^2 - 1 \quad \text{(by archimedian prop)}$$

$$n_0 + 1 > (M+1)^2$$

$$\sqrt{n_0+1} - 1 > M$$

Thus, S_n is a unbounded above.

Thus, $\sum_{k=1}^{\infty} a_k = \lim_{n \to \infty} \sum_{k=1}^{n} a_k$

$$= \lim_{n \to \infty} S_n$$

is divergent. (thm 3.24)

b) $$a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n}$$

$$a_n = \frac{(\sqrt{n+1} - \sqrt{n})}{n(\sqrt{n+1} + \sqrt{n})} \times (\sqrt{n+1} + \sqrt{n})$$

$$a_n = \frac{\cancel{n}+1-\cancel{n}}{n(\sqrt{n+1}+\sqrt{n})}$$

$$a_n = \frac{1}{n(\sqrt{n+1}+\sqrt{n})} \leq$$

$$|a_n| = \left|\frac{1}{n(\sqrt{n+1}+\sqrt{n})}\right| = \frac{1}{n(\sqrt{n+1}+\sqrt{n})}$$

$$|a_n| = \frac{1}{n(\sqrt{n+1}+\sqrt{n})} \leq \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$$

for all $n \in \mathbb{Z}^+$

$\cancel{|a_n| \leq \frac{1}{n^{3/2}}}$

~~By com~~ Note that $\sum \frac{1}{n^{3/2}}$ is cgt. (thm 3.28)

By comparsion test (thm 3.25) ~~is~~ $\sum a_n$ is cgt also.

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c) $a_n = (\sqrt[n]{n} - 1)^n$

We are going to use root test.

put $\alpha = \limsup_{n \to \infty} \sqrt[n]{|a_n|}$

$$\alpha = \limsup_{n \to \infty} \sqrt[n]{|(\sqrt[n]{n} - 1)^n|}$$

$$\alpha = \limsup_{n \to \infty} |(\sqrt[n]{n} - 1)|$$

Note that $\lim_{n \to \infty} |\sqrt[n]{n} - 1| = |\lim_{n \to \infty} \sqrt[n]{n} - \lim_{n \to \infty} 1|$

$= |1 - 1| = 0$ (thm 3.20)

Then

$$1 > \cancel{\leq} \ \alpha = \limsup_{n \to \infty} \sqrt[n]{n} - 1 = \lim_{n \to \infty} \sqrt[n]{n} - 1 = 0$$

(example 3.18)

Thus, $\cancel{\lim_{n \to \infty}}$ $\sum a_n$ is a cgt series.

d) Let $z \in \mathbb{C}$

$$a_n = \frac{1}{1+z^n}$$

Case-I If $|z| \leq 1$. Then,

$$|1+z^n| \leq |1| + |z^n| \quad \text{(trinangle inequalit}$$
$$= 1 + |z^n|$$
$$= 1 + |z|^n$$
$$\leq 1 + 1^n = 2 \quad \text{(by hypothesis)}$$

Thus, $\frac{1}{2} \leq \frac{1}{|1+z^n|}$

$\lim_{n\to\infty} a_n \neq 0$. Therefore,

$\sum a_n$ is diverges (by thm 3.23)

Case-II If $|z| > 1$, ~~the~~

Then, $|z^n| = |1+z^n - 1| \leq |1+z^n| + |-1|$ (by triangle inequality)

$$= |1+z^n| + 1$$

Thus, $|z^n| - 1 < |1+z^n|$ ———— (*)

First observe following,

claim: $\sum \frac{1}{|z|^n - 1}$ is cgt

We use ratio test,

$$\lim_{n \to \infty} \left| \frac{1/|z|^{n+1} - 1}{1/|z|^n - 1} \right| = \lim_{n \to \infty} \frac{|z|^n - 1}{|z|^{n+1} - 1}$$

$$= \lim_{n \to \infty} \frac{1}{|z|} \frac{\left(1 - \frac{1}{|z|^n}\right)}{\left(1 - \frac{1}{|z|^{n+1}}\right)} = \frac{1}{|z|} < 1$$

$\sum$ is cgt.

by Comparison test and claim and ⊛,

~~$\sum a_n = \sum$~~

$$|a_n| = \left| \frac{1}{1 + z^n} \right| \leq \frac{1}{|z^n - 1|} \text{ is cgt.}$$

7. Prove that the convergence of Σa_n implies the convergence of

$$\Sigma \frac{\sqrt{a_n}}{n},$$

if $a_n \geq 0$.

Suppose that $a_n \geq 0$ for all $n \in \mathbb{Z}^+$

First, recall AM-GM inequality

$$\frac{x+y}{2} \geq \sqrt{xy}$$ for non-negtive x, y

$$\sqrt{a_n}\,\frac{1}{n} \leq \frac{a_n + 1/n^2}{2} \quad \text{for all } n \in \mathbb{Z}^+$$

Now consider,

$$\Sigma \frac{(a_n + 1/n^2)}{2} = \frac{1}{2} \Sigma (a_n + 1/n^2)$$

Note that $\Sigma 1/n^2$ is convergent.

If a_n is convergent then $\Sigma(a_n + 1/n^2)$ is also convergent. Then $\Sigma \frac{(a_n + 1/n^2)}{2}$ is also convergent.

By comparison test, a_n is cgt $\Rightarrow$ $\Sigma \sqrt{a_n}/n$ is cgt.

8. If Σa_n converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\Sigma a_n b_n$ converges.

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8) If Σa_n cgs and if $\{b_n\}$ is monotonic and bounded.

~~first notethat~~ $\{b_n\}$ is cgt. ~~(By thm 3.14)~~

~~Let say~~ $\lim_{n\to\infty} b_n = \ell_b$ —(1)

~~Rough work~~

I think we can use Thm 3.42.

Let $A_n = \sum_{k=0}^{n} a_k$

by thm 3.24

claim 1: $\{A_n\}$ is boanded.

Σa_n is cgt $\Rightarrow \lim_{n\to\infty} A_n$ is cgt.

let's say $\lim_{n\to\infty} A_n = \ell_a$

by def^{n} of convergence,

$\exists N_0$, $n \geq N_0 \Rightarrow |A_n - \ell_a| < 1$

$n \geq N_0 \Rightarrow ||A_n| - |\ell_a|| < 1$

$n \geq N_0 \Longrightarrow |A_n| - |\ell_a| < 1 < -|A_n| + |\ell_a|$

$n \geq N_0 \Longrightarrow |A_n| < 1 + |\ell_a|$

Let $M = \max\{|A_1|, |A_2|, \dots, |A_{N_0}|, |l_a|+1\}$

Since this is finite set ~~there existan~~ maximum exists.

Thus, $|A_n| \leq M$ forall $n \in \mathbb{N}$

Therefore $\{A_n\}$ is a bounded sequnce.

So, it statify 1st condition on thm 3.42.

~~Let $C_n =$~~

Since $\{b_n\}$ is monotone and bdd it is cgt. (by thm 3.14). Let say

$$\lim_{n \to \infty} b_n = b$$

Case-1 : If $b_{n+1} \geq b_n$ for all $n \in \mathbb{N}^*$

$$b_{n+1} \geq b_n$$
$$b_{n+1} - b \geq b_n - b$$

Let $c_n = \cancel{(b_n - b)} = (b - b_n)$

Case-II If $b_{n+1} \leq b_n$ for all $n \in \mathbb{N}^+$

$$\Rightarrow b_{n+1} - b \leq b_n - b$$

$$(b - b_{n+1}) \geq (b - b_n)$$

Let $c_n = -\cancel{(b - b_n)} = (b_n - b)$

Both case observe that

$$c_0 \geq c_1 \geq c_2 \geq \cdots \geq c_n \geq c_{n+1} \geq \cdots$$

So, it statify 2nd condition thm 3.42.

Further,

$$\lim_{n \to \infty} b_n = b$$

$$\lim_{n \to \infty} \underbrace{(b_n - b)}_{} = 0$$

$$\lim_{n \to \infty} c_n = 0$$

So, it statify 3rd condition statify the 3.42. Then $\sum a_n c_n$ is cgt

$$\sum a_n(b_n - b) = \sum a_n b_n - \sum a_n b$$

~~$\sum a_n(b_n-b) + \sum b a_n = \sum a_n b_n$~~ -

$$\sum a_n b_n = \sum a_n(b_n - b) + \sum a_n b$$

$$= \sum a_n(b_n - b) + b \sum a_n$$

Since $\sum a_n(b_n - b)$ and $\sum a_n$ are cgt, $\sum a_n b_n$ also cgt.

$\blacksquare$

9. Find the radius of convergence of each of the following power series:

(*a*) $\sum n^3 z^n$, (*b*) $\sum \frac{2^n}{n!} z^n$,

(*c*) $\sum \frac{2^n}{n^2} z^n$, (*d*) $\sum \frac{n^3}{3^n} z^n$.

$\boxed{\text{Thm}}$ Given Powerseries, $\sum_n C_n z^n$

Let $\alpha = \limsup_{n \to \infty} \frac{C_n}{C_{n+1}}$ $R = \frac{1}{\alpha}$

$\left(\begin{array}{l} \text{If } \alpha = 0, \text{ then } R = +\infty \\ \text{If } \alpha = +\infty \text{ then } R = 0 \end{array} \right)$

If $|z| < R$ then $\sum_n C_n z^n$ is cgt.

If $|z| > R$ then $\sum_n C_n z^n$ is dgt.

proof: Let $a_n = C_n z^n$ apply the root test.

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{C_{n+1} z^{n+1}}{C_n z^n} \right| = \left| \frac{C_{n+1}}{C_n} \right| |z|$$

$$\limsup_{n\to\infty}\left|\frac{C_{n+1}Z^{n+1}}{C_n Z^n}\right| = \limsup_{n\to\infty} |Z| \left|\frac{C_{n+1}}{C_n}\right|$$

$$= |Z| \limsup_{n\to\infty} \left|\frac{C_{n+1}}{C_n}\right| = \frac{|Z|}{R}$$

If $|Z| < R$ then $\limsup_{n\to\infty}\left|\frac{C_{n+1}Z^{n+1}}{C_n Z^n}\right| < 1$

Hence, by ratio test. $\sum C_n Z^n$ is cgt

If $|Z| > R$ then $\limsup_{n\to\infty}\left|\frac{C_n Z^{n+1}}{C_n Z^n}\right| > 1$

Hence ratio test

$\sum C_n Z^n$ is dgt.

a) We are going ratio test

$$\lim_{n\to\infty} \frac{C_{n+1}}{C_n} = \cancel{(n+1)^3} \lim_{n\to\infty} \frac{(n+1)^3}{n^3}$$

$$= \lim_{n\to\infty} (1+1/n)^3 = 1 \Rightarrow \boxed{R=1}$$

b) $\sum (2^n/n!) z^n$

$$\lim_{n\to\infty} \frac{2^{n+1}/(n+1)!}{2^n/n!} = \lim_{n\to\infty} \frac{2^1}{(n+1)} = 0$$

$$= \cancel{\lim_{n\to\infty} \frac{2}{1+1}}$$ Thus $\boxed{R=+\infty}$

c) $\sum \frac{2^n}{n^2} z^n$

$$\lim_{n\to\infty} \frac{2^{n+1}/(n+1)^2}{2^n/(n^2)} = \lim_{n\to\infty} \frac{2n^2}{(n+1)^2}$$

$$= \lim_{n\to\infty} \frac{2}{\left(1+\frac{1}{n}\right)^2} = 2 \Rightarrow \boxed{R = 1/2}$$

d) $\sum n^3/3^n$

$$\lim_{n\to\infty} \frac{(n+1)^3/3^{n+1}}{n^3/3^n} = \cancel{(n+1)^3} \lim_{n\to\infty} \frac{1}{3}\frac{(n+1)^3}{n^3}$$

$$= \frac{1}{3}\lim_{n\to\infty} \frac{(1+1/n)^3}{1} = \frac{1}{3} \Rightarrow \boxed{R = 3}$$

10. Suppose that the coefficients of the power series $\sum a_n z^n$ are integers, infinitely many of which are distinct from zero. Prove that the radius of convergence is at most 1.

(10) Suppose that the coefficients of power series, $\sum a_n z^n$ are integers

i.e. $a_n \in \mathbb{Z}$ for all $n \in \mathbb{Z}$.

$a_i \neq 0$ infinetly many $i \in \mathbb{N}$

NTS: R = radius of convergence ≤ 1

We are going to proove that, If $|z| > 1$ then $\sum a_n z^n$ is divergent.

Suppose that $|z| > 1$ —— (1)

$$|a_n z^n| = |a_n| |z|^n > |a_n|$$

there are infinetly many $m \in \mathbb{Z}$ such tat

$|a_m| \geq 1$ ($\because \exists$ infinety many $m \in \mathbb{Z}$ such that $a_m \neq 0$)

Thus $\lim_{n \to \infty} a_n \neq 0$. Thus $\sum a_n$ is dgt

(thm 3.23)

11. Suppose $a_n > 0$, $s_n = a_1 + \cdots + a_n$, and Σa_n diverges.

(*a*) Prove that $\sum \frac{a_n}{1 + a_n}$ diverges.

(*b*) Prove that

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$$

and deduce that $\sum \frac{a_n}{s_n}$ diverges.

(*c*) Prove that

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

and deduce that $\sum \frac{a_n}{s_n^2}$ converges.

(*d*) What can be said about

$$\sum \frac{a_n}{1 + na_n} \quad \text{and} \quad \sum \frac{a_n}{1 + n^2 a_n}?$$

Exercise 3.11

a) ~~Case 1 If a_n does not bounded,~~ ~~then~~ $\frac{a_n}{1+a_n}$

Assume the contradiction tha $\sum \frac{a_n}{1+a_n}$ converges. Then by thm 3.23,

$$\lim_{n \to \infty} \frac{a_n}{1+a_n} = 0$$

$$\lim_{n \to \infty} \frac{1}{\left(1+\frac{1}{a_n}\right)} = 0 \quad (\text{Since } a_n \neq 0)$$

$$\Rightarrow \lim_{n \to \infty} \frac{1}{a_n} = \infty$$

$$\Rightarrow \lim_{n \to \infty} a_n = 0$$

Then $\forall \varepsilon > 0 \ \exists N_0 \in \mathbb{Z}^+$ such that $\forall n \in \mathbb{N}$, & $n \geq N_0 \Rightarrow |a_n - 0| < \varepsilon$

We can find, $N_0 \in \mathbb{Z}^+$ such that $\forall n \in \mathbb{N}$

$$n \geq N_0 \Rightarrow |a_n| < 1$$

$$\Rightarrow \quad a_n < 1 \quad (\text{Since } a_n > 0$$

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* $$a_n < 1$$
$$1 + a_n < 2$$

$$\frac{a_n}{1+a_n} \geq \frac{a_n}{2}$$

~~By comparison test,~~
~~Let rewrite test~~
Let me summrize this,
$\exists N_0 \in \mathbb{Z}^+$ such that $\forall n \in \mathbb{N}$

$$n \geq N_0 \implies \frac{a_n}{1+a_n} \geq \frac{a_n}{2}$$

By comparison test. (thm 3.25 a)

$\sum \frac{a_n}{2}$ is cgt.

$\implies \sum a_n$ is also cgt.

This is ~~a~~ contion. Thus $\sum a_n/(1+a_n)$ is dgt.

b) First note that S_n is monotonically increas[ing]

$$\left(\begin{array}{l} \text{Since } a_n > 0 \text{ for all } n \in \mathbb{Z}^+ \\ a_1 + \cdots + a_n < \underbrace{\cancel{a_1} + \cdots + a_n + a_{n+1}} \\ \cancel{\Rightarrow}\ S_n < S_{n+1} \text{ for all } n \in \mathbb{N} \end{array} \right)$$

~~Thus,~~ $\cancel{S_N < S_{N+1}}$

Second note that:

$$a_N = S_N - S_{N-1}$$

$$\frac{a_{N+1}}{S_{N+1}} + \cdots + \frac{a_{N+k}}{S_{N+k}} \geq \frac{a_{N+1} + \cdots + a_{N+k}}{S_{N+k}}$$

$$\cancel{\geq} = \frac{S_{N+k} - S_N}{S_{N+k}}$$

$$= \cancel{S}\, 1 - \frac{S_N}{S_{N+k}}$$

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Assume the contradiction, $\Sigma a_n/s_n$ conveges,

First of all note that, s_n is monotoic increase and unbounded. (If otherwise Σa_n is cgt.)

Thus,

$\forall n \in \mathbb{N} \ \exists k_0 = k_0(n) \in \mathbb{Z}^+$ such that

$$2S_n \leq S_{n+k}$$

$$\frac{S_n}{S_{n+k}} \leq \frac{1}{2} \quad \text{—— (1)}$$

By thm 3.22

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$,

$$l_1 \geq l_2 \geq N \Rightarrow \left| \sum_{i=l_1}^{l_2} \frac{a_i}{s_i} \right| < \varepsilon$$

$$\Rightarrow \left| \frac{a_{l_1}}{s_{l_1}} + \cdots + \frac{a_{l_2}}{s_{l_2}} \right| < \varepsilon$$

(Since $a_n > 0$ and $s_n > 0$ for all $n \in \mathbb{N}$)

$$\Rightarrow 0 < \frac{a_{l_1}}{s_{l_1}} + \cdots + \frac{a_{l_2}}{s_{l_2}} < \varepsilon$$

for $l_1 = N+1$ and $l_2 = N+k_0$

$$\frac{1}{2} = 1 - \frac{1}{2} \leq 1 - \frac{S_N}{S_{N+k_0}} < \frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k_0}}{s_{N+k_0}} < \varepsilon$$

This contrdict the fact ε is abitary small.

Therefore, $\sum a_n/s_n$ divegers.

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c) $$\frac{1}{S_{n-1}} - \frac{1}{S_n} = \frac{S_n - S_{n-1}}{S_n S_{n-1}} = \frac{a_n}{S_n S_{n-1}}$$

$$\geq \frac{a_n}{S_n^2}$$

(Since $S_n \leq S_{n+1}$)

Claim: $\sum_{n=2}^{\infty} \left(\frac{1}{S_{n-1}} - \frac{1}{S_n} \right)$ is cgt to $\frac{1}{a_1}$

---

first of all observe that

$$\frac{a_n}{S_n^2} \leq \frac{1}{S_{n-1}} - \frac{1}{S_n} \quad \text{for all } n = 2, 3, 4 \ldots$$

$$\sum_{n=1}^{N} \frac{a_n}{S_n^2} = \frac{a_1}{S_1^2} + \sum_{n=2}^{N} \frac{a_n}{S_n^2}$$

$$\leq \frac{a}{S_1^2} + \sum_{n=2}^{N} \left(\frac{1}{S_{n-1}} - \frac{1}{S_n} \right)$$

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$$\sum_{n=2}^{N}\left(\frac{1}{S_{n-1}}-\frac{1}{S_n}\right)$$

$$=\left(\frac{1}{S_1}-\cancel{\frac{1}{S_2}}\right)+\left(\cancel{\frac{1}{S_2}}-\cancel{\frac{1}{S_3}}\right)+\cdots+\left(\cancel{\frac{1}{S_{N-2}}}-\cancel{\frac{1}{S_{N-1}}}\right)+\left(\cancel{\frac{1}{S_{N-1}}}-\frac{1}{S_N}\right)$$

$$=\left(\frac{1}{S_1}-\frac{1}{S_N}\right)$$

$$\sum_{n=1}^{N}\frac{a_n}{S_n^2}\leq\frac{a_1}{S_1^2}+\frac{1}{S_1}-\frac{1}{S_N}$$

$$\lim_{N\to\infty}\sum_{n=1}^{N}\frac{a_n}{S_n^2}\leq\frac{a_1}{S_1^2}+\frac{1}{S_1}-\lim_{N\to\infty}\frac{1}{S_N}$$

$$\sum_{n=1}^{\infty}\frac{a_n}{S_n^2}\leq\frac{a_1}{S_1^2}+\frac{1}{S_1}=\frac{a_1}{a_1^2}+\frac{1}{a_1}=\frac{2}{a_1}$$

(Note that $S_1=a_1$)

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Thus, $\sum_{n=1}^{\infty} a_n/s_n^2$ is bdd. above

Note that $\sum_{n=1}^{N} a_n/s_n^2$ is monotically incr

Because

$$\frac{a_1}{S_1^2} + \cdots + \frac{a_N}{S_N^2} \leq \frac{a_1}{S_1^2} + \cdots + \frac{a_N}{S_N^2} + \frac{a_{N+1}}{S_{N+1}^2}$$

(Since $a_n > 0$ and $S_n > 0$)

Thus $\sum a_n/S_n^2$ is convergent.

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d) Now we consider, $\sum \frac{a_n}{1+n^2a_n}$

First observe that,

$$\frac{a_n}{1+n^2a_n} \leq \frac{a_n}{n^2a_n} = \frac{1}{n^2}$$

We know that $\sum 1/n^2$ is cgt. By comparison test $\sum a_n/(1+n^2a_n)$ is also cgt.

Second we consider series $\sum a_n/(1+na_n)$

This series may be convergent or divergent

It competly depends on the term a_n

Divergent.

Let $a_n = 1/n$ for all $n \in \mathbb{Z}^+$

a_n is divergent

Then

$$\frac{a_n}{1+na_n} = \frac{1}{2n}$$

we know that $\sum 1/n$ is divergent.

Then $\sum \frac{1}{2n}$ is also divergent.

Convergent

Let

$$a_n = \begin{cases} 1/\sqrt{n} & \text{if } n = 1, 4, 9 \\ \frac{1}{n^2} & \text{if otherwise} \end{cases}$$

Case-1 If n is perfect number

$n = k^2$ for some $k \in \mathbb{Z}^+$

$$a_n = \frac{1}{k}$$

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$$\frac{a_n}{1+na_n} = \frac{1/k}{1+k^2/k} = \frac{1}{k+k^2} \leq \frac{1}{k^2}$$

Case-II If is n is perfect square

Then $a_n = 1/n^2$

So, $\frac{a_n}{1+na_n} = \frac{1/n^2}{1+n/n^2} = \frac{1}{n^2+n} \leq \frac{1}{n}$

Then consider, partial sum

$$\sum_{n=1}^{N} \frac{a_n}{1+na_n} = \cancel{\sum_{n=1,4,9,16,\ldots}}$$

$$\leq \sum_{\substack{1\leq n\leq N \\ n=m^2, m\in\mathbb{Z}^+}} \frac{1}{n} + \sum_{\substack{1\leq n\leq N \\ n\neq m^2, m\in\mathbb{Z}}} \frac{1}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{a_n}{1+na_n} \leq \sum_{m=1}^{\infty} \frac{1}{m^2} + \sum_{n=1}^{\infty} \frac{1}{n^2}$$

~~Thus~~ righside is ~~bounded~~ convergent

Therefore by comparison test $\sum \frac{a_n}{1+na_n}$ is divergent.

*Ask from sir!

(12) Suppose that $a_n > 0$ and $\sum a_n$ converges.

Put $r_n = \sum_{m=n}^{\infty} a_m$

~~a) Proove that~~

a) Suppose that $m < n$

First note that, r_n is monotonically ~~increasing~~ decreasing.

$$\underbrace{a_n + a_{n+1} + a_{n+2} + \cdots}_{r_n} \geq \underbrace{a_{n+1} + a_{n+2} + \cdots}_{r_{n+1}} \quad \text{for all } n \in \mathbb{Z}^+ \quad (\text{since } a_n > 0)$$

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} \geq \frac{a_m}{r_m} + \cdots + \frac{a_n}{r_m} = \frac{a_m + \cdots + a_n}{r_m}$$

$$\cancel{= \frac{r_m - r_{n+1}}{r_m} = 1 - \frac{r_{n+1}}{r_m}}$$

$$> \frac{a_m + \cdots + a_{n-1}}{r_m} = \frac{r_m - r_n}{r_m}$$

$$= 1 - \frac{r_n}{r_m}$$