Therefore, by (*) and (**),

$$\overline{B_n} = \overline{\left(\bigcup_{j=1}^{n} A_j\right)} = \bigcup \overline{A_j}$$

b)

Let $A_1, A_2, \ldots$ be subsets of a metric space. Suppose $B = \bigcup_{i=1}^{\infty} A_i$

$$A_i \subseteq \overline{A_i} \text{ for all } i = 1, 2, \ldots$$

$$B = \bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{i=1}^{\infty} \overline{A_i}$$

$$A_j \subseteq \bigcup_{i=1}^{\infty} A_i \subseteq \overline{\left(\bigcup_{i=1}^{\infty} A_i\right)} = \overline{B}$$

for all $j = 1, 2, \ldots$

$$A_j \subseteq \overline{B} \text{ for all } j = 1, 2, \ldots$$

Thus, $\bigcup_{j=1}^{\infty} \overline{A_j} \subseteq \overline{B}$

We know that rational numbers are countable. Then we can write $\mathbb{Q}$ as follow

$$\mathbb{Q} = \{r_1, r_2, r_3, \dots\} =$$

Let $A_i = \{r_i\}$ for all $i = 1, 2, \dots$

$$\overline{A_i} = A_i \cup A_i' = A_i \cup \emptyset \text{ (limit points of finite set is empty)}$$

$$\overline{A_i} = A_i$$

Now observe, $\bigcup_{i=1}^{\infty} A_i = \mathbb{Q}$

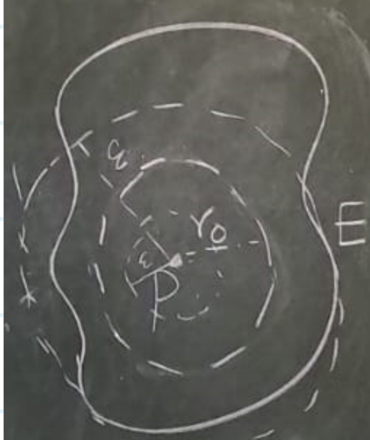
We know that $\overline{\mathbb{Q}} = \mathbb{R}$

But $\bigcup_{i=1}^{\infty} \overline{A_i} = \bigcup_{i=1}^{\infty} A_i = \mathbb{Q} \neq \mathbb{R} = \overline{\mathbb{Q}} = \overline{\left(\bigcup_{i=1}^{\infty} A_i\right)}$

8. Is every point of every open set $E \subset R^2$ a limit point of E? Answer the same question for closed sets in R^2.

Open set in $\mathbb{R}^2$

Yes. In forevery point in every open set is a limit point.



Let $p=(x_1,y_1)\in E\subseteq\mathbb{R}^2$

$\exists\, r_0>0$ such that

$$N_{r_0}(p)\subseteq E \quad \text{——} \; (1)$$

Let $\varepsilon>0$.

choose $\tilde{r}=\min\{\varepsilon, r_0\}>0$.

$$d\left((x_1,y_1),\left(x_1+\frac{\tilde{r}}{2},y_1\right)\right)=\frac{\tilde{r}}{2}<\tilde{r}<r_0$$

Thus $\left(x_1+\frac{\tilde{r}}{2},y_1\right)\in N_{r_0}(p)\subseteq E$ (By (1)) —— (2)

Furthur, $d((x_1+\frac{\tilde{r}}{2}, y_1),(x_1, y_1)) = \frac{\tilde{r}}{2} < \tilde{r} \leq \varepsilon$

Thus, $(x_1+\frac{\tilde{r}}{2}, y_1) \in N_\varepsilon(P)$ — ③

By ③ and ②, $\forall \varepsilon > 0\ \exists q \in E$ such that $p \neq q$ and $q \in N_\varepsilon(P)$.

(Here $q = (x_1+\frac{\tilde{r}}{2}, y_1) \neq (x_1+\frac{\tilde{r}}{2}, y_1) = P$ ($\because \tilde{r} > 0$))

Closed set in $\mathbb{R}^2$.

Let $E=\{0\}$, but $E'=\phi$

Therefore, given statement for closed set is not TRUE.

9. Let E° denote the set of all interior points of a set E. [See Definition 2.18(*e*); E° is called the *interior* of E.]

(*a*) Prove that E° is always open.

(*b*) Prove that E is open if and only if $E^\circ = E$.

(*c*) If $G \subset E$ and G is open, prove that $G \subset E^\circ$.

(*d*) Prove that the complement of E° is the closure of the complement of E.

(*e*) Do E and $\bar{E}$ always have the same interiors?

(*f*) Do E and E° always have the same closures?

Let E° be the set all of all interior points

NTS: $(E^\circ) \subseteq (E^\circ)^\circ$

Let $p \in E^\circ$. Then $\exists r > 0$ such that $N_r(p) \subseteq E$

let $q \in N_r(p) \subseteq E$. Now let $\varepsilon = r - d(p,q) > 0$

Let $s \in N_\varepsilon(q)$.

$d(p,s) \leq d(p,q) + d(q,s) < r$

$s \in N_r(p) \subseteq E$

Hence, $N_\varepsilon(q) \subseteq E$

Therefore $\exists \varepsilon > 0$ s.t. $N_\varepsilon(q) \subseteq E$

Hence $q \in E^\circ$

Since q is arbitary $N_r(p) \subseteq E^\circ$

So, $\exists r > 0$ such that $N_r(p) \subseteq E^\circ$,

$p \in (E^\circ)^\circ$.

Therefore, $(E^\circ) \subseteq (E^\circ)^\circ$

By part (a) we get, If $E^{\circ} = E$ then E is open. —— (*)

Now suppose that E is open. Therefore we know that $E \subseteq E^{\circ}$ —— (1)

Let $p \in E^{\circ}$. Then $\exists r > 0$ such that $N_r(p) \subseteq E$. That means $p \in E$.

Therefore $E^{\circ} \subseteq E$ —— (2)

By (1) and (2), $E^{\circ} = E$. Hence If E is open then $E = E^{\circ}$ —— (**)

c) Suppose that $G \subset E$ and G is open

Since G is open. by part b) $G = G^\circ$

First, ~~prove that~~ we are going proove that ~~This is~~ following claim

Claim: Let A, B two sets ~~in metric space~~. Suy

$A \subseteq B \implies A^\circ \subseteq B^\circ$

Suppose that $A \subseteq B$.

Let $x \in A^\circ$. Then there exist ~~a~~ $r > 0$ such that $N_r(x) \subseteq$ [illegible] $A \subseteq B$. Therefore $x \in B^\circ$. Thus $A^\circ \subseteq B^\circ$

Since $G = G^\circ \subseteq E^\circ$. Therefore, $G \subseteq E^\circ$

$\blacksquare$

Need to show: $(E^{\circ})^c \subseteq = \overline{(E^c)}$

<u>Claim1:</u> $(E^{\circ})^c \subseteq \overline{(E^c)}$

By $\text{def}^{\underline{n}}$ of closure (2.26 def^n)

$$E^c \subseteq \overline{(E^c)}$$

Then, $E = (E^c)^c \supseteq \left(\overline{(E^c)}\right)^c$

Note that $\overline{(E^c)}$ is closed. ($\because$ Thm 2.27 a)

Then $\left(\overline{(E^c)}\right)^c$ is ~~cl~~ open ($\because$ Thm 2.23)

Then part b),

$$E^{\circ} \supseteq \left(\overline{(E^c)}\right)^c$$

Then, $(E^{\circ})^c \subseteq \overline{(E^c)}$

Claim 2: $(E^{o})^{c} \supseteq \overline{(E^{c})}$

~~First note that $\overline{(E^{c})}$ is closed.~~

~~Then~~ By definition closure ($\text{Def}^{\underline{n}}$ 2.26)

$$(E^{c}) \subseteq \overline{(E^{c})} \quad \text{—} \quad 1$$

By $\text{def}^{\underline{n}}$ 2.18, we get $E^{o} \subseteq E$

Then, $(E^{o})^{c} \supseteq E^{c}$ — 2

Since E^{o} is open, then $(E^{o})^{c}$ is closed — 3

(by ~~def~~ thm 2.23)

Since closure is the smallest closed subset that contains (E^{c}). (by thm 2.27)

$$\overline{(E^{c})} \subseteq (E^{o})^{c}$$

By claim 1 and 2, $\overline{(E^{c})} = (E^{o})^{c}$

Need to show: $(E^{\circ})^c \subseteq \overline{(E^c)}$

<u>Claim 1:</u> $(E^{\circ})^c \subseteq \overline{(E^c)}$

By def$^{\underline{n}}$ of Closure (2.26 def$^{\underline{n}}$)

$$E^c \subseteq \overline{(E^c)}$$

Then, $E = (E^c)^c \supseteq \left(\overline{(E^c)}\right)^c$

Note that $\overline{(E^c)}$ is closed. ($\because$ Thm 2.27 a)

Then $\left(\overline{(E^c)}\right)^c$ is ~~cl~~ open ($\because$ Thm 2.23)

Then part b),

$$E^{\circ} \supseteq \left(\overline{(E^c)}\right)^c$$

Then, $(E^{\circ})^c \subseteq \overline{(E^c)}$

10. Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p, q) = \begin{cases} 1 & (\text{if } p \neq q) \\ 0 & (\text{if } p = q). \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

10. (M1) It is very clear that ~~pc~~ ~~d~~ for $p \in X, q \in X$

$p(p,q) \geq 0$ and $d(p,q) = 0$ iff $p = q$.

(M2) It is obvious also $d(p,q) = d(q,p)$

(M3) Here non-trival part is the triangle inequlity.

Let $p, q, r \in X$.

N.T.S: $d(p, r) \leq d(p,q) + d(q,r)$

Case-1 If $p = r$,

$d(p,r) = 0 \leq d(p,q) + d(q,r)$ (1st property of def^n of metric ~~spce~~)

Case-II If $p \neq r$

Then $d(p,r)=1$. Now ~~assume~~ triangle inqequlity fails if $p=q=r$. ~~So, assume~~ $p=r$
If it happen $p=q$ and $q=r$ implies ~~$p\neq r$~~.
~~p~~ this contriddict assumption $p \neq r$
Therefore at least one of following is true.

$$p \neq q \quad \text{or} \quad q \neq r$$

Therefore triangle inequity holds.

Redo

It!

11. For $x \in R^1$ and $y \in R^1$, define

$$d_1(x, y) = (x - y)^2,$$
$$d_2(x, y) = \sqrt{|x - y|},$$
$$d_3(x, y) = |x^2 - y^2|,$$
$$d_4(x, y) = |x - 2y|,$$
$$d_5(x, y) = \frac{|x - y|}{1 + |x - y|}.$$

Determine, for each of these, whether it is a metric or not.

Let $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

① $d_1(x,y)$ is NOT a metric. Because it fails the triangle inequility.

$d_1(0,1) + d_1(1,2) = (0-1)^2 + (1-2)^2 = 1+1 = 2$

$d_1(0,2) = (0-2)^2 = 4$

Thus,

$d_1(0,1) + d_1(1,2) < d_1(0,2)$

② $d_2(\cancel{x,y})$ is a Metric.

(M1) It is trivial that $\sqrt{|x-y|} \geq 0$ and

$$\sqrt{|x-y|}=0 \text{ iff } x=y$$

(M2) Furthar this ~~case~~ condition also trivial.

$$d_2(x,y)=\sqrt{|x-y|}=\sqrt{|y-x|}=d_2(y,x)$$

(M3) The nontrival part is trianglein–

$$(d_2(x,y))^2 = |x-y| \leq |x-z| + |z-y|$$
$$\leq |x-z| + |z-y| + 2\sqrt{|x-z|}\sqrt{|y-z|}$$
$$= \left(\sqrt{|x-z|} + \sqrt{|z-y|}\right)^2$$
$$= (d(x,z) + d(z,y))^2$$

Therefore, triangle inequality holds.

(3) $d_3(x,y)$ is <u>not</u> a metric

$$d_3(1,-1) = |\cancel{2}1^2 - (-1)^2| = |1-1| = 0$$

So this fails, $d_3(x,y) = 0$ iff $x = 0$

(4) $d_4(x,y)$ is <u>not</u> a metric.

$$d_4(0,1) = |0 - 2(1)| = 2 \neq 1 = |1 - 2(0)| = d_4(1,0)$$

$d_s(\cancel{x,y})$ is ametric.

Clearly Since $|x-y| \geq 0$

$$d_s(x,y) = \frac{|x-y|}{1-|x-y|} \geq 0$$

Furthur,

$$d_s(x-y) = 0 \iff \frac{|x-y|}{1-|x-y|} = 0$$

$$\iff |x-y| = 0$$

$$\iff x = y.$$

~~Furthur:~~

Since. $|x-y) = |y-x|$

$$d_s(x,y) = \frac{|x-y|}{1+|x-y|}$$

$$= \frac{|y-x|}{1+|y-x|}$$

$$= d_s(y,x)$$

Let ~~x~~ $x, y, z \in \mathbb{R}$. Let

$p = \cancel{d} |x - z|$

$q = |x-y|$

$r = |y-z|$

By tringle inequlity of $|\cdot|$ in $\mathbb{R}$

$$|x-z| \leq |x-y| + |y-z|$$

$$\cancel{\leq} p \leq q + r$$

Then
$$p \leq q + r + 2qr + pqr$$

(Since $p, q, r \geq 0$)

So,

$$p + pq + pr + pqr \leq q + r + pq + pr + 2qr + 2pqr$$

Therefore, $\dfrac{p}{1+p} \leq \dfrac{q}{1+q} + \dfrac{r}{1+r}$

~~Ther~~ Hence $d_s(\cdot, \cdot)$ holds tringle inequlity.

12. Let $K \subset R^1$ consist of 0 and the numbers $1/n$, for $n = 1, 2, 3, \ldots$. Prove that K is compact directly from the definition (without using the Heine-Borel theorem).

(12) Let $K = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} \subseteq \mathbb{R}$

Let $\{G_\alpha\}_{\alpha \in I}$ be an opencover of K.

Then $0 \in G_{\alpha_0}$ forsome $\alpha_0 \in I$

Since G_{α_0} is open, by def$^{\underline{n}}$ ~~of open~~ 2.18)
$\exists r > 0$ s.t. $N_r(0) \subseteq G_{\alpha_0}$

By archmiedian property, ~~there exist an~~
$\exists N \in \mathbb{Z}^+$, $\frac{1}{N} < r$

Thus, if $n \geq N$ then

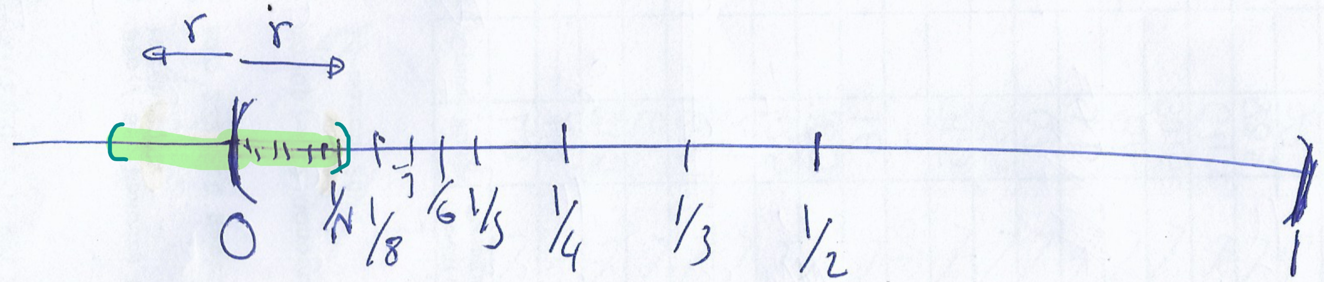
$$\frac{1}{n} \leq \frac{1}{N} < r$$

Thus, $\frac{1}{n} \in N_r(0) \subseteq G_{\alpha_0}$ if $n \geq N$

Similarly thre exist some opensets G_{α_i} suchthat

$$\frac{1}{i} \in G_{\alpha_i} \quad \text{for } i = 1, 2, \ldots, N$$

Therefore, $K \subseteq \bigcup_{i=0}^{N} G_{\alpha_i}$


r
r
0
1/N
1/8
1/7
1/6
1/5
1/4
1/3
1/2
1

19. (*a*) If A and B are disjoint closed sets in some metric space X, prove that they are separated.

(*b*) Prove the same for disjoint open sets.

(*c*) Fix $p \in X$, $\delta > 0$, define A to be the set of all $q \in X$ for which $d(p, q) < \delta$, define B similarly, with $>$ in place of $<$. Prove that A and B are separated.

(*d*) Prove that every connected metric space with at least two points is uncountable. *Hint*: Use (*c*).

(19) Let X be an metric space. Suppose that $A, B \subseteq X$ with $A \cap B = \emptyset$.

First suppose A, B are closed.

a) Since A, B are closed. Then $\overline{A} = A, \overline{B} = B$ (thm 2.27). Thus,

$$\overline{A} \cap B = A \cap B = \emptyset = A \cap B = A \cap \overline{B}$$

Thus A and B are separeted

b) Now suppose that A, B are ~~c~~ open.

B is open $\Rightarrow$ B^c is closed

Since $A \cap B = \emptyset$, $B^c \supseteq A$. Then,

~~B^c~~ By thm 2.27, $B^c \supseteq \overline{A}$

This implies $\overline{A} \cap B = \emptyset$

Similarly we can show that

$A \cap \overline{B} = \emptyset$.

Therefore A, B are seprated.

c) Fix $p \in X$ and $\delta > 0$. Let

$A := \{q \in X \mid d(p,q) < \delta\}$ and

$B := \{q \in X \mid d(p,q) > \delta\}$.

It is very clear that $A \cap B = \emptyset$.

Then by part b), A and B are separted.

(Becase A and B are open).