1. Prove that the empty set is a subset of every set.

Let A bean set. We are going to use proof by contradiction. So, assume that

$$\emptyset \not\subset A$$

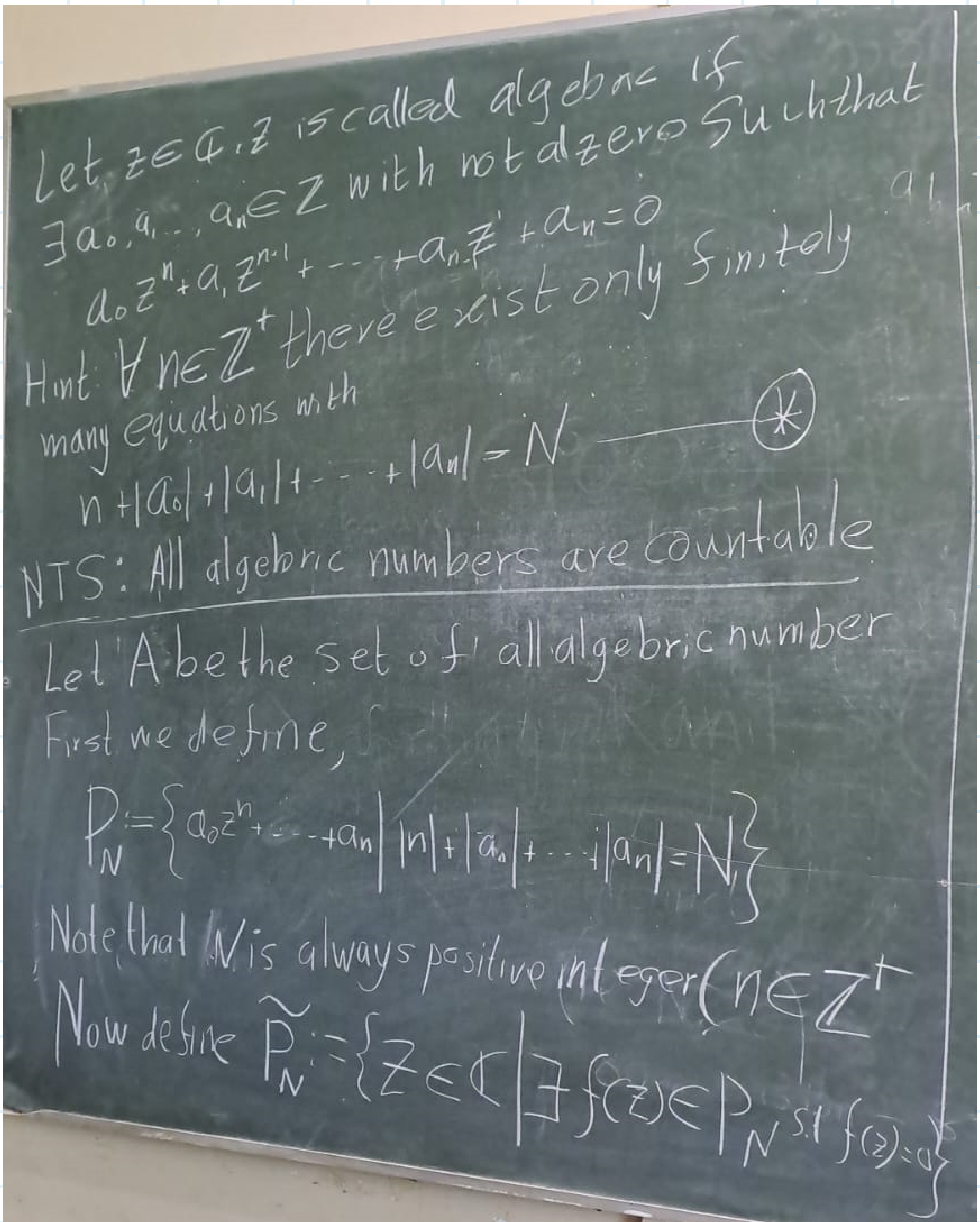
That means thre exist $x \in \emptyset$ such that $x \notin A$. This is contradiction because $\emptyset$ is empty. Therefore, ~~A is empt~~ $\emptyset$ is asubset of A.

2. A complex number z is said to be *algebraic* if there are integers $a_0, \ldots, a_n$, not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$

Prove that the set of all algebraic numbers is countable. *Hint:* For every positive integer N there are only finitely many equations with

$$n + |a_0| + |a_1| + \cdots + |a_n| = N.$$



We know that n-degree polynomimal have atmost n-roots

Then $|\tilde{P}_N| < \infty$

Now observe that $A = \bigcup_{N=1}^{\infty} \tilde{P}_N$

By thm 2.12 we get A is countable.

Now we are going to prove that n-degree polynomial have at most n-roots.

Let $f(z) = \sum_{k=0}^{n} a_k z^k$

We are going use mathematical induction

When $n=0$, It is trivial

Assume that $n-1$ degree polynomials have at most $n-1$ roots

We know that n-degree polynomimal have atmost n-roots

Then $|\tilde{P}_N| < \infty$

Now observe that $A = \bigcup_{N=1}^{\infty} \tilde{P}_N$

By thm 2.12 we get A is countable.

3. Prove that there exist real numbers which are not algebraic.

Assume the contray that all
real numbers are algebric. In part 2)
We proved that, All algebric numbers
are countable. Therefore, it contradict
the fact that $\mathbb{R}$ are not countable.
Therefore, our assumption is false.
Hence there are some real numbers
that are <u>not</u> algebric.

4. Is the set of all irrational real numbers countable?

(4) No.
Assume set of irrational numbers are countable, we know that rational numbers are countable. Then $\mathbb{R} = (\mathbb{Q} \cup \mathbb{R} \setminus \mathbb{Q})$ are countable. This is a contradiction.
Therefore $\mathbb{R} \setminus \mathbb{Q}$ are uncountable.

5. Construct a bounded set of real numbers with exactly three limit points.

$A = \left\{ \frac{1}{n}, 2+\frac{1}{n}, 4+\frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\}$

Claim $A' = \{0, 2, 4\}$.

Let $N_r(0)$ be an nbd around 0.

By archimidean property, we can find $n \in \mathbb{N}$ such that $\frac{1}{n} < r$. Then $0 \neq \frac{1}{n} \in N_r(0)$. Therefore 0 is an limit point. $\left(\frac{1}{n} \in A\right)$

Now we are going to show that

Let $N_r(2)$ be an neighbourhood around 2.

By archimedian property, We can find $m \in \mathbb{N}$ such that

$$2 + \frac{1}{m} < r + 2.$$

Furthur, $2 \neq 2 + \frac{1}{m} \in N_r(2)$ and $2 + \frac{1}{m} \in A$

We can apply the same way to show that 4 is also an limit point.

Now, We are gonna to show that there do <u>not</u> exist any other limit point. Let $x \in A \setminus \{0, 2, 4\}$

(Our choice is based on the all limit point are in the A)

All the elements in the A in the form

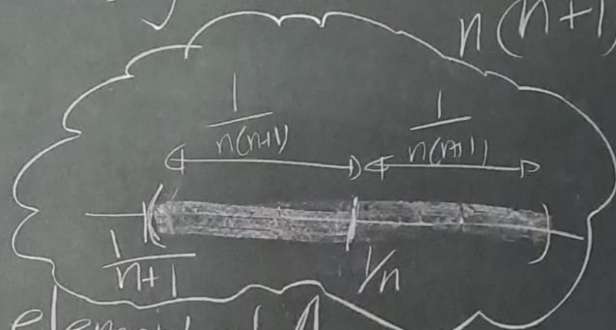
either $\frac{1}{n}$ or $2+\frac{1}{n}$ or $4+\frac{1}{n}$ for some $n\in\mathbb{N}$

First consider $\frac{1}{n}$ form.

Let $y\in A$ such that $y=\frac{1}{n}$ for some $n\in\mathbb{N}$.

Now consider Neighbourhood of y wit radius $\frac{1}{n(n+1)}$

$N_{\frac{1}{n(n+1)}}(y)$.



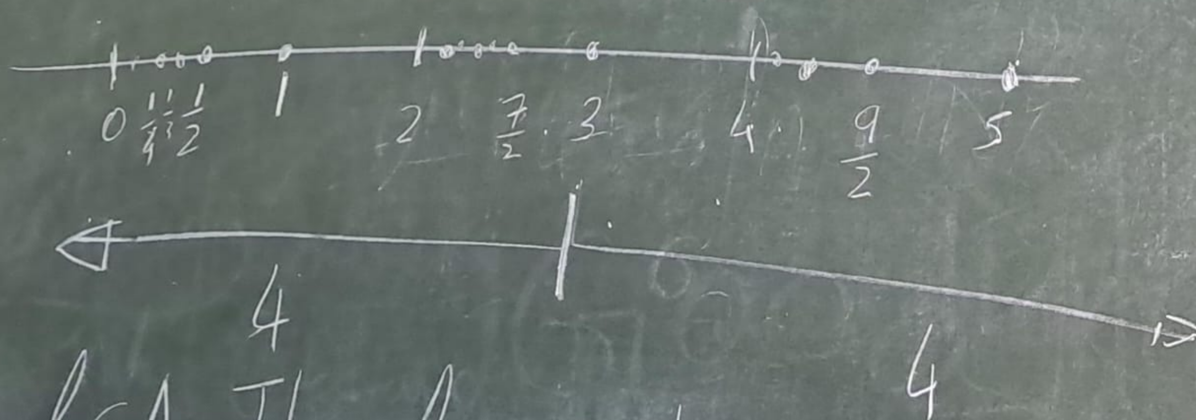
It does not contain any element of A.

$\left(k\in A \Rightarrow k\notin N_{\frac{1}{n(n+1)}}(y)\right)$. Thus $y=\frac{1}{n}\notin A'$

Similarly, using the other form also, we can get same result, Therefore there is no any other limit points other than 0, 2, 4.

Finally, we have to show that;

A is bounded.



Let $l \in A$. Then $l = p + \frac{1}{n}$, $p = 0, 2, 4$
for some $n \in \mathbb{N}$.

$$d(3,l) = \left|3 - \left(p + \frac{1}{n}\right)\right| \leq |3 - p| + \left|\frac{1}{n}\right| \quad (\text{Since } p \text{ either } 0, 2, 4)$$

$$\leq 3 + \left|\frac{1}{n}\right| \leq 4 < 5$$

Therefore, A is bdd.

6. Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\bar{E}$ have the same limit points. (Recall that $\bar{E} = E \cup E'$.) Do E and E' always have the same limit points?

Let p be a limit point of E' $(p \in (E')')$

$\forall r > 0\ \exists q_0 \in E'$ such that $d(p, q_0) < r/2$

Since $q_0 \in E'$,

$\forall r > 0\ \exists q_1 \in E$ such that $d(q_0, q_1) < r/2$

Then,

$\forall r > 0\ \exists q_1 \in E$ such that,

$$d(p, q_1) \leq d(p, q_0) + d(q_0, q_1) < \frac{r}{2} + \frac{r}{2} = r$$

Thus, $p \in E'$. Thus, $(E')' \subseteq E'$

Therefore E' is closed (by def^{n})

Let X be an metric space and $A, B \subseteq X$

If $A \subseteq B$ then $A' \subseteq B'$

Proof: Let $a \in A'$. $\forall r > 0\ \exists b \in A \setminus \{P\}$ such that

$$d(a,b) < r.$$

Since $A \subseteq B$ then

$\forall r > 0\ \exists b \in B \setminus \{P\}$ such that $d(a,b) < r$.

Thus, $b \in B'$. Hence $A' \subseteq B'$.

Therefore, $A \subseteq B \Rightarrow A' \subseteq B'$.

NTS: $(\bar{E})' = E'$

Claim1: $E' \subseteq (\bar{E})'$

Let $p \in E'$, then $p \in E' \subseteq \bar{E}$. Thus, $p \in (\bar{E})'$

Hence $E' \subseteq (\bar{E})'$.

Claim2: $(\bar{E})' \subseteq E'$

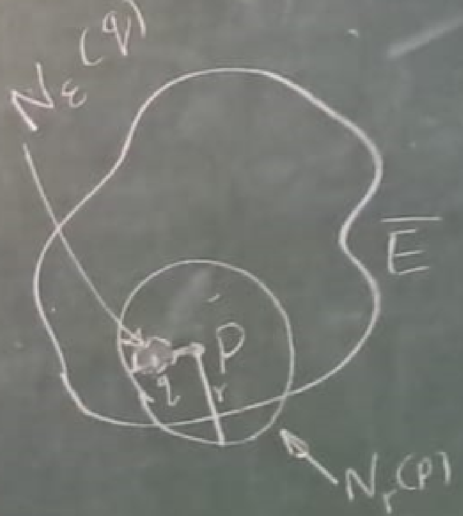
Let $p \in (\bar{E})'$. Let $r > 0$. Then $\exists q \in \bar{E}$ $p \neq q$ such that $q \in N_r(p)$. Since $q \in \bar{E} = E \cup E'$

$q \in E$ or $q \in E'$

If $q \in E$, then, since r is arbitrary, we can write this as, $\forall r > 0\ \exists q \in E$ such that $p \neq q$ and $q \in N_r(p)$

If $q \in E'$, Now chose

$\varepsilon = \min\{r - d(p,q),\ d(p,q)\} > 0$



Since $q \in E'$ Then there exist $s \in E$ such that,

$s \in N_\varepsilon(q)$ and $s \neq q$

It is very clear that $N_\varepsilon(q) \subseteq N_r(p)$.

Note that $s \neq p$, (by choice of ε)

Therefore $\forall r > 0\ \exists s \in E$ s.t. $p \neq s$ such that $s \in N_r(p)$. Thus $p \in E'$. Hence $(\bar{E})' \subseteq E'$

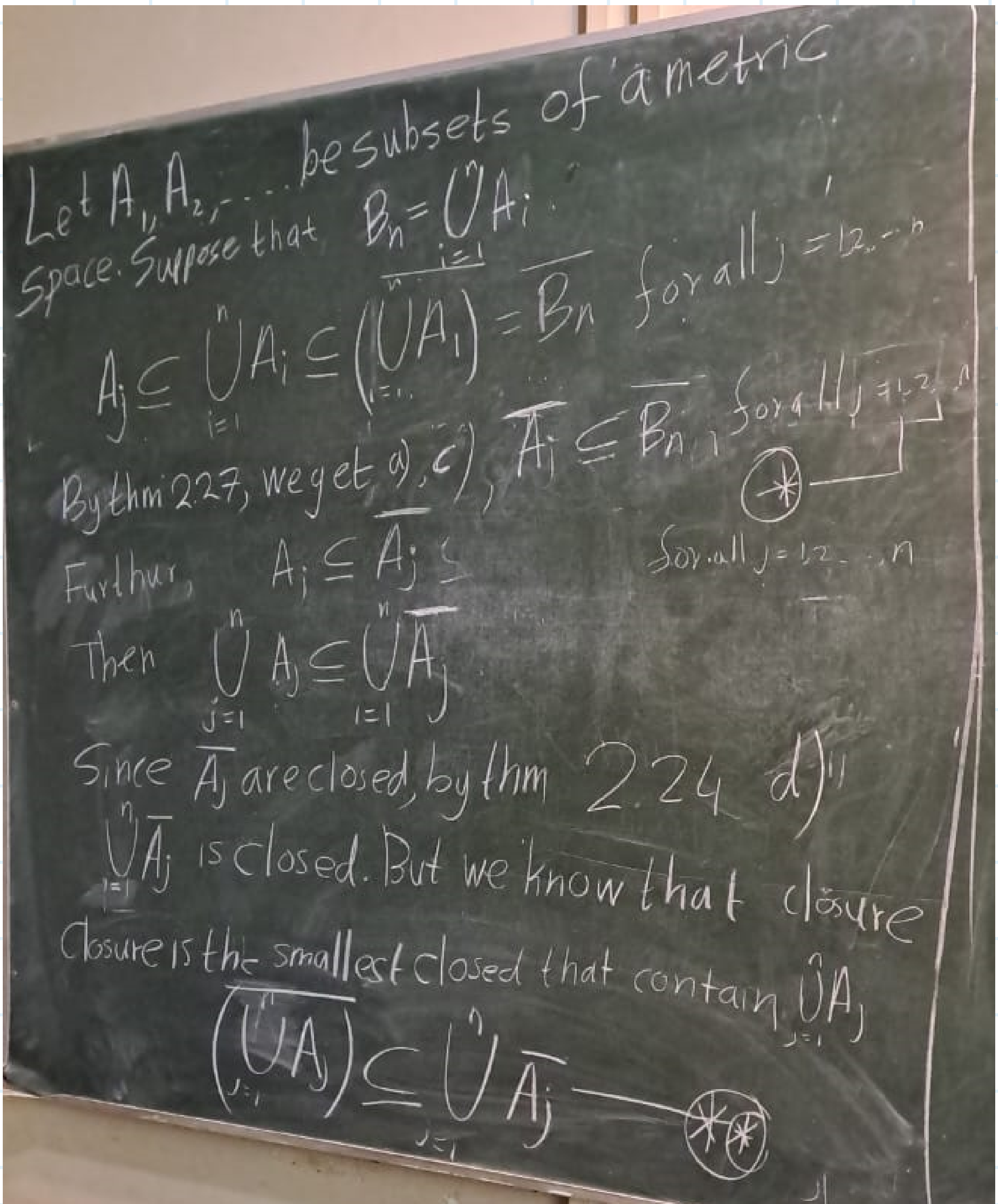
Therefore, $(\bar{E})' = E'$

7. Let $A_1, A_2, A_3, \ldots$ be subsets of a metric space.

(*a*) If $B_n = \bigcup_{i=1}^{n} A_i$, prove that $\bar{B}_n = \bigcup_{i=1}^{n} \bar{A}_i$, for $n = 1, 2, 3, \ldots$.

(*b*) If $B = \bigcup_{i=1}^{\infty} A_i$, prove that $\bar{B} \supset \bigcup_{i=1}^{\infty} \bar{A}_i$.

Show, by an example, that this inclusion can be proper.



Therefore, by (*) and (**),

$$\overline{B_n} = \overline{\left(\bigcup_{j=1}^{n} A_j\right)} = \bigcup \overline{A_j}$$