15. **Under what conditions does equality hold in the Schwarz inequality?**

Recall the Schwarz inequality

1.35 Theorem *If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are complex numbers, then*

$$\left|\sum_{j=1}^{n} a_j \bar{b}_j\right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2.$$

Proof Put $A = \Sigma|a_j|^2$, $B = \Sigma|b_j|^2$, $C = \Sigma a_j \bar{b}_j$ (in all sums in this proof, j runs over the values $1, \ldots, n$). If $B = 0$, then $b_1 = \cdots = b_n = 0$, and the conclusion is trivial. Assume therefore that $B > 0$. By Theorem 1.31 we have

$$\begin{aligned}\sum |Ba_j - Cb_j|^2 &= \sum (Ba_j - Cb_j)(B\bar{a}_j - \overline{Cb_j}) \\ &= B^2 \sum |a_j|^2 - B\bar{C} \sum a_j \bar{b}_j - BC \sum \bar{a}_j b_j + |C|^2 \sum |b_j|^2 \\ &= B^2 A - B|C|^2 \\ &= B(AB - |C|^2).\end{aligned}$$

Since each term in the first sum is nonnegative, we see that

$$B(AB - |C|^2) \geq 0.$$

Since $B > 0$, it follows that $AB - |C|^2 \geq 0$. This is the desired inequality.

If we go through the proof, we can See that, If $A=0$ or $B=0$ implies ~~that~~ the equility. (This case is trivial)

~~Now~~ So, now assume that $A\neq 0$ and $B\neq 0$.

By the prooof thm 1.35

$\cancel{(AB-|C|^2)=0}$

(equlity of Schwartz inequlity) $\Longleftrightarrow \sum |Ba_j - Cb_j|^2 = 0$

$\Longleftrightarrow |Ba_j - Cb_j| = 0$ forall $i=1,2,\dots,n$

$\Longleftrightarrow Ba_j = Cb_j$ forall $i=1,2\dots n$

$\Longleftrightarrow a_j = \frac{C}{B} b_j$ forall $i=1,2\dots n$

$\Longleftrightarrow$ the numbers a_j are propostional to b_j

$\Longleftrightarrow$ a_j and b_j are linearly dependent.

16. Suppose $k \geq 3$, $\mathbf{x}, \mathbf{y} \in R^k$, $|\mathbf{x} - \mathbf{y}| = d > 0$, and $r > 0$. Pr

(*a*) If $2r > d$, there are infinitely many $\mathbf{z} \in R^k$ such that

$$|\mathbf{z} - \mathbf{x}| = |\mathbf{z} - \mathbf{y}| = r.$$

(*b*) If $2r = d$, there is exactly one such $\mathbf{z}$.

(*c*) If $2r < d$, there is no such $\mathbf{z}$.

How must these statements be modified if k is 2 or 1?

17. Prove that

$$|\mathbf{x}+\mathbf{y}|^2+|\mathbf{x}-\mathbf{y}|^2=2|\mathbf{x}|^2+2|\mathbf{y}|^2$$

if $\mathbf{x}\in R^k$ and $\mathbf{y}\in R^k$. Interpret this geometrically, as a statement about parallelograms.

Suppose that $x, y \in \mathbb{R}^k$. Here $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$

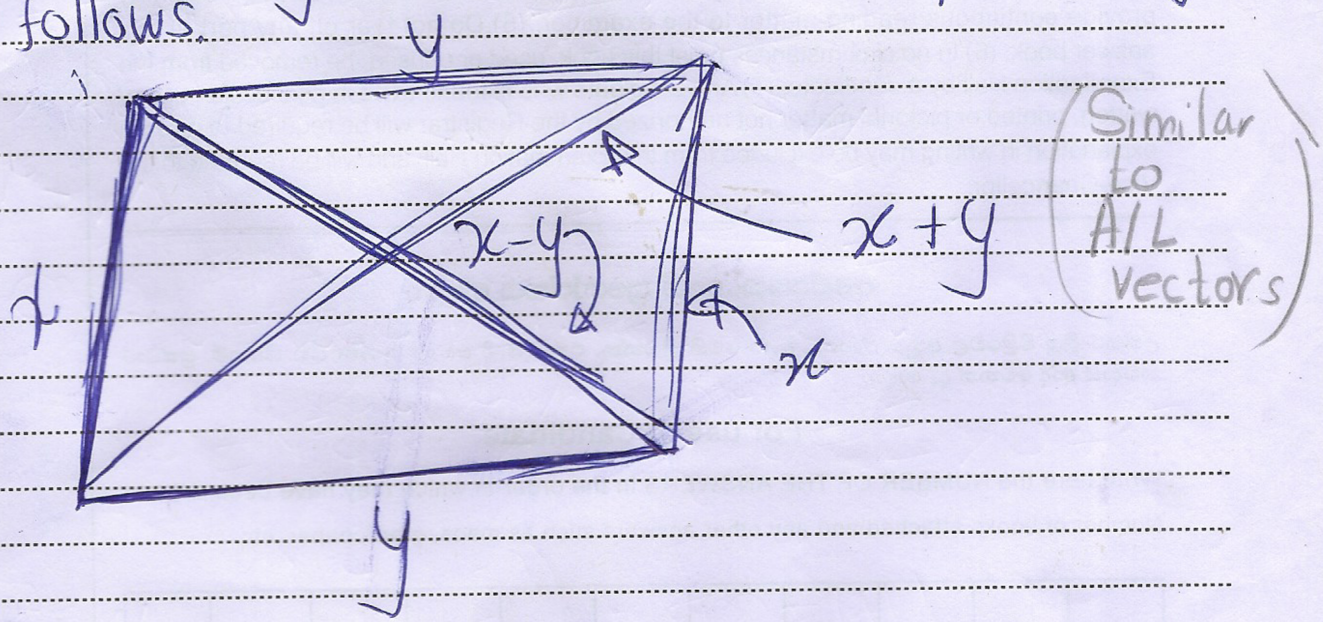
$$|x+y|^2 = \big((x+y)\cdot(x+y)\big)$$
$$= x\cdot x + 2(x\cdot y) + y\cdot y$$
$$= |x|^2 + 2(x\cdot y) + |y|^2 \quad \text{——— (1)}$$

$$|x-y|^2 = (x-y)\cdot(x-y)$$
$$= x\cdot x - 2(x\cdot y) + y\cdot y$$
$$= |x|^2 - 2(x\cdot y) + |y|^2 \quad \text{——— (2)}$$

By (1) and (2)

$$|x+y|^2 + |x-y|^2 = |x|^2 + 2(x\cdot y) + |y|^2 + |x|^2 - 2(x\cdot y) + |y|^2$$
$$= 2|x|^2 + 2|y|^2$$

If x and y are sides of parallelgram as follows.



Then $x-y$ and $x+y$ are diangongols of the parallelgram. Therefore we can conclude that;

The sum of square of diagonals of parallelgram is equal to sum of square of the sides.

18. If $k \geq 2$ and $\mathbf{x} \in R^k$, prove that there exists $\mathbf{y} \in R^k$ such that $\mathbf{y} \neq \mathbf{0}$ but $\mathbf{x} \cdot \mathbf{y} = 0$. Is this also true if $k = 1$?

Suppose that $k \geq 2$ and $x \in \mathbb{R}^k$

Here $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$

- If $x = 0$, then we can ~~get~~ ~~what~~ choose any nonzero vector.

- If ~~some~~ $x_i = 0$ for some $i = 1, 2, \ldots, k$

but not all component zero ($x \neq 0$)

Then we can choose $y = (y_1, \ldots, y_k)$ as

$y_i = 1$ and other every component equal to 0. Then y is nonzero. Further

$x \cdot y = 0$

eg: $(x_1, x_2, \ldots, \underset{n}{x}, 0, \underset{n+2}{x}, \ldots x_k) \cdot (0, \ldots, 1, \ldots 0)$

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$= x_1 \cdot 0 + x_2 \cdot 0, \ldots, 0 \cdot 1, x_{n+2} \cdot 0 + \cdots + x_k 0$

$= 0$

- If all component of x are nonzero

$$x_i = 0 \quad \text{for all} \quad i = 1, 2, \ldots, k$$

We can choose $y = (-x_2, x_1, 0, \ldots, 0)$
Then $y \neq 0$.

$$x \cdot y$$
$$= (x_1, x_2, \ldots, x_n) \cdot (-x_2, x_1, 0, \ldots, 0)$$
$$= (-x_1 x_2 + x_2 x_1 + 0 + \cdots + 0)$$
$$= 0$$

By proposition 1.6-b.), The given statement doesnot hold when $k = 1$

19. Suppose $\mathbf{a} \in R^k$, $\mathbf{b} \in R^k$. Find $\mathbf{c} \in R^k$ and $r > 0$ such that

$$|\mathbf{x} - \mathbf{a}| = 2|\mathbf{x} - \mathbf{b}|$$

if and only if $|\mathbf{x} - \mathbf{c}| = r$.
(*Solution:* $3\mathbf{c} = 4\mathbf{b} - \mathbf{a}$, $3r = 2|\mathbf{b} - \mathbf{a}|$.)

Suppose that $a, b \in \mathbb{R}^k$. We are going to use ~~that~~ the given answers.

First observe that
~~Forwa~~

$$x \cdot y = \sum_{i=1}^{k} x_i y_i = \sum_{i=1}^{k} y_i x_i = y \cdot x$$

(Since x_i, y_i is real number)

$$|x-a| = 2|x-b| \iff |x-a|^2 = 4|x-b|^2$$
$$\iff (x-a)\cdot(x-a) = 4(x-b)\cdot(x-b)$$
$$\iff x\cdot x - 2a\cdot x + a\cdot a = 4(x\cdot x - 2b\cdot x + b\cdot b)$$
$$\iff |x|^2 - 2(a\cdot x) + |a|^2 = 4|x|^2 - 8(b\cdot x) + 4|b|^2$$
$$\iff 3|x|^2 - 8(b\cdot x) + 2(a\cdot x) - |a|^2 + 4|b|^2 = 0$$
$$\iff |x|^2 - \frac{8}{3}(b\cdot x) + \frac{2}{3}(a\cdot x) - \frac{1}{3}|a|^2 + \frac{4}{3}|b|^2 = 0$$
$$\iff |x|^2 - \frac{8}{3}(b\cdot x) + \frac{2}{3}(a\cdot x) + \frac{16}{9}|b|^2 + \frac{1}{9}|a|^2 - \frac{8}{9}(b\cdot a) \left(+ \frac{8}{9}(b\cdot a) - \frac{4}{9}|b|^2 - \frac{4}{9}|a|^2 \right) = 0$$
$$\iff \left|x - \frac{4}{3}b + \frac{1}{3}a\right|^2 - \frac{4}{9}|b-a|^2 = 0$$
$$\iff \left|x - \frac{4}{3}b + \frac{1}{3}a\right|^2 = \frac{2}{3}|b-a|^2$$

$$\Longleftrightarrow \quad |x-c|^2 = 4|x-b|^2$$

$$\Longleftrightarrow \quad |x-c| = 2|x-b|$$

20) Redo It