8. Prove that no order can be defined in the complex field that turns it into an ordered field. *Hint:* -1 is a square.

Assume that $\mathbb{C}$ is one orderd feild.

~~Then (by prop 1.18) Since~~ $i \neq 0$. We consider i

Then by definition 1.5, one of them is true.

$$i < 0, \quad i = 0, \quad i > 0.$$

We know that i is non-zero. So we consider two cases $i < 0$ and $i > 0$

Case-I $i < 0$

Then, $-i > 0$ (by Prop 1.18)

$$(-i)(-i) = (i)(i) = i^2 = -1 > 0$$

(by def^n 1.17-ii)

This is contradiction.

Case-II $i > 0$. Then

$$(i)(i) = i^2 = -1 > 0 \text{ (by def}^n\text{ 1.17-ii)}$$

This is also contradiction.

Therefore, there is no order relation in $\mathbb{C}$

9. Suppose $z = a + bi$, $w = c + di$. Define $z < w$ if $a < c$, and also if $a = c$ but $b < d$. Prove that this turns the set of all complex numbers into an ordered set. (This type of order relation is called a *dictionary order*, or *lexicographic order*, for obvious reasons.) Does this ordered set have the least-upper-bound property?

Let $z = a+bi$ and $w = c+di$ where $a, b, c, d \in \mathbb{R}$. Since $\mathbb{R}$ is anorderd set.

$(a<c$ or $a=c$ or $\cancel{b<a}$ $a>c)$
and $(b<d$ or $b=d$ or $b>d)$

<u>Case-I</u> If $a<c$, then $z<w$

<u>Case-II</u> If $a>c$ then $z>w$

<u>Case-III</u> If $a=c$ and $b>d$ then $z>w$

<u>Case-IV</u> If $a=c$ and $b<d$ then $z<w$

<u>Case-V</u> If $a=c$ and $b=d$ then $z=w$

Therefore one of them always true. ie: either $z<w$ or $z=w$ or $z>w$

Now let's check second condion of sorderd set

Let $u = e+fi$.

~~Suppose that $z>w$ and $y>0$~~

Suppose that $z<w$ and $w<u$

$z<w \Rightarrow (a<c)$ or $(a=c$ and $b<d)$

$w < u \Rightarrow (c < e)$ or $(c = e$ and $d < f)$

Case-I If $a < c$ and $c < e$ then $a < e$

Thus, $z < u$

Case-II If $a < c$ and $c = e$ and $d < f$ then $a < e$. This implies $z < u$.

Case-III If $a = c$ and $b < d$ and $c < e$ then $a < e$ then $z < u$

Case-IV If $a = c$ and $b < d$ and $c = e$ and $d < f$. Then $a = e$ and $b < f$.

This implies $z < u$.

This 4 cases implies ~~$z < w$ and $w < u$~~

If $z < w$ and $w < u$ then $z < w$.

10. Suppose $z = a + bi$, $w = u + iv$, and

$$a = \left(\frac{|w| + u}{2}\right)^{1/2}, \qquad b = \left(\frac{|w| - u}{2}\right)^{1/2}.$$

Prove that $z^2 = w$ if $v \geq 0$ and that $(\bar{z})^2 = w$ if $v \leq 0$. Conclude that every complex number (with one exception!) has two complex square roots.

Suppose that $z = a + bi$ and $w = u + iv$ and ~~where~~ $a = \left(\frac{|w|+u}{2}\right)^{1/2}$, $b = \left(\frac{|w|-u}{2}\right)^{1/2}$

Suppose that $v \geq 0$.

Need to show: $z^2 = w$

$$z^2 = (a+bi)^2 = (a+bi)(a+bi)$$
$$= a^2 + bi + bi + 2i^2 ab$$
$$= (a^2 - b^2) + 2iab$$

$$a^2 - b^2 = \left(\frac{|w|+u}{2}\right) + \left(\frac{|w|-u}{2}\right) = u$$

$$2ab = 2\left(\frac{|w|+u}{2}\right)^{1/2}\left(\frac{|w|-u}{2}\right)^{1/2} = 2\left(\left(\frac{|w|+u}{2}\right)\left(\frac{|w|-u}{2}\right)\right)^{1/2}$$
$$= 2\left(\frac{|w|^2 - u^2}{4}\right)^{1/2} = 2\left(\frac{v^2}{4}\right)^{1/2} = v$$

(Since $v \geq 0$)

Thus,

$$z^2 = (a^2 - b^2) + (2ab)i$$
$$= u + vi = w$$

NTS: If $\gamma \leq 0$ then $(\overline{z})^2 = w$

Suppose that $\gamma \leq 0$.

$$\overline{z}^2 = \overline{z}\,\overline{z} = (a-bi)(a-bi)$$
$$= a - bi - bi + i^2 b$$
$$= (a^2 - b^2) - (2ab)i$$

We already show that $(a^2-b^2) = u$ and

$$2ab = 2\left(\frac{\gamma^2}{2}\right)^{1/2}.$$

Since $\gamma \leq 0$, $2ab = 2\left(\frac{\gamma^2}{2}\right)^{1/2} = -\gamma$

Therefore, $(\overline{z})^2 = (a^2-b^2) - (2ab)i$

$$= u - (-\gamma)i$$
$$= u + \gamma i = w$$

11. If z is a complex number, prove that there exists an $r \geq 0$ and a complex number w with $|w| = 1$ such that $z = rw$. Are w and r always uniquely determined by z?

Suppose that $z \in \mathbb{C}$.

- If $z = 0$, choose $r = 0$ and $w = \frac{1}{\sqrt{2}} + \left(\frac{1}{\sqrt{2}}\right)i$

Then $|w| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$

- If $z \neq 0$, choose $r = |z|$ and $w = \frac{z}{|z|}$. Then $r = |z| \geq 0$ and

$$|w| = \left|\frac{z}{|z|}\right| = \frac{|z|}{||z||} = \frac{|z|}{|z|} = 1$$

(by thm 1.33-c)

(by thm 1.33d)

(This idea is get from the unit vectors)

Therefore, $z \in \mathbb{C} \Rightarrow (\exists (r \geq 0$ and $w \in \mathbb{C}$ with $|w| = 1)$ such that $z = rw$

- In firs case, uniqunes does not holds. Because if $z=0$, then we can choose $r=0$ and any $w \in \mathbb{C}$ with $|w|=1$.

- In second case, uniquness holds. If $z \neq 0$ then we can chose

Assume that $z = r_1 w_1 = r_2 w_2$ — ① with $r_1 > 0$, $r_2 > 0$, $|w_1| = 1$, $|w_2| = 1$. Take modulus of both sides

$$|r_1 w_1| = |r_2 w_2|$$

$$|r_1||w_1| = |r_2||w_2| \quad (\text{thm } 1.33 \text{ and}$$

$$r_1 |w_1| = r_2 |w_2| \quad (\text{Since } r_i \in \mathbb{R})$$

$$r_1 = r_2 \text{ — ②} \quad (\text{Since } |w_1| = |w_2| = 1$$

Thus, ~~$r_1 w_1 = r_2 w_2$~~ By ~~our~~ equation ①, ② impiles $w_1 = w_2$

Therefore r, w are uniquely determined by z.

12. If $z_1, \ldots, z_n$ are complex, prove that

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|.$$

(12) Suppose that $Z_1, Z_2, \ldots, Z_n \in \mathbb{C}$

We are going to use mathematical inducti

$\cancel{|Z_1 + Z_2 + \cdots + Z_n| =}$

If $n=2$, by thm 1.33 trinangle equlity holds. i.e:

$$|Z_1 + Z_2| \leq |Z_1| + |Z_2|$$

Now assume that following holds for $n = p \in \mathbb{N}$

$$|Z_1 + Z_2 + \cdots + Z_p| \leq |Z_1| + \cdots + |Z_p|$$

Now Consider following,

$$|Z_1 + Z_2 + \cdots + Z_p + Z_{p+1}| \leq |Z_1 + \cdots + Z_p| + |Z_{p+1}|$$

(by base case)

$$|Z_1 + \cdots + Z_p + Z_{p+1}| \leq |Z_1| + \cdots + |Z_p| + |Z_{p+1}|$$

(Induction hypothesis)

Therefore, by mathematical hypothesis, ~~we get~~

$$|Z_1 + Z_2 + \cdots + Z_n| \leq |Z_1| + \cdots + |Z_n| \text{ for all } n \in \mathbb{N} \setminus \{0\}$$

13. If x, y are complex, prove that

$$||x| - |y|| \leq |x - y|.$$

Let $x, y \in \mathbb{C}$. By theorm 1.33

$$|z+w| \leq |z| + |w| \text{ for all } z, w \in \mathbb{C}$$

We use $z = (x-y)$ and $w = y$. Using triangle inequlity,

$$|(x-y)+y| \leq |x-y| + |y|$$

$$|x| \leq |x-y| + |y|$$

$$|x| - |y| \leq |x-y| \quad \text{———} \; (*)$$

Now, we use $z = y-x$ and $w = x$. By triangle inequlity,

$$|(y-x)+x| \leq |y-x| + |x|$$

$$|y| \leq |y-x| + |x|$$

$$|y| - |x| \leq |y-x|$$

We know that, $|x-y| = |y-x|$. Thus,

$$|y| - |x| \leq |x-y| \quad \text{———} \; (**)$$

By $(*)$ and $(**)$, $||x| - |y|| \leq |x-y|$

$\blacksquare$

14. If z is a complex number such that $|z| = 1$, that is, such that $z\bar{z} = 1$, compute

$$|1 + z|^2 + |1 - z|^2.$$

Suppose that $z \in \mathbb{C}$ such that $|z| = 1$.

P By definit 1.32,

$$
\begin{aligned}
|1+z|^2 &= (1+z)\overline{(1+z)} && (\text{def}^n\ 1.32)\\
&= (1+z)(\bar{1}+\bar{z}) && (\text{By thm } 1.31\text{-a})\\
&= (1+z)(1+\bar{z}) && (\because 1 \in \mathbb{R})\\
&= 1+\bar{z}+z+z\bar{z} && (\text{distributi property})\\
&= 1+\bar{z}+z+|z| && (\text{def}^n\ 1.32)\\
&= 1+\bar{z}+z+1 && (\text{by hypothesis})\\
&= 2+\bar{z}+z && \text{—— (1)}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|1-z|^2 &= (1-z)(1-\bar{z}) && (\text{def}^n\ 1.32 \text{ and thm } 1.31)\\
&= 1-\bar{z}-z+z\bar{z} && (\text{distributive})\\
&= 1-\bar{z}-\bar{z}+|z| && (\text{def}^n\ 1.32)\\
&= 1-\bar{z}-\bar{z}+1 && (\text{Hypothesis})\\
&= 2-\bar{z}+\bar{z} && \text{—— (2)}
\end{aligned}
$$

By (1) and (2),

$|1+z|^2 + |1-z|^2 = 4$