Exercise 1.1 If r is rational $r \neq 0$ and x is irrational, prove that $r + x$ and rx are irrational.

Suppose that $0 \neq r \in \mathbb{Q}$ and $x \in \mathbb{R} \setminus \mathbb{Q}$

- **Claim 1**: $r + x$ is irrational.
  Assume the contray that $r + x \in \mathbb{Q}$. Since, $\mathbb{Q}$ is field,
  $$x = (r + x) - r \in \mathbb{Q}$$
  . This is a contradiction. Thus, $r + x$ is irrational.
- **Claim 2**: rx is irrational.
  Assume the contray that $rx \in \mathbb{Q}$. Since, $\mathbb{Q}$ is field,
  $$\left(\frac{1}{x}\right) rx = \left(\frac{1}{x}\right) xr = \left(\frac{1}{x} x\right) r = x \in \mathbb{Q}$$
  . This is a contradiction. Thus, rx is irrational.

Exercise 1.2 (1:R2) Prove that there is no rational number whose square is 12.

Proof. First observe that

$$\sqrt{12} = \sqrt{4 \cdot 3} = 2\sqrt{3}$$

Claim 1 : $\forall n \in \mathbb{N} 3|n^2 \implies 3|n$.
Let $n \in \mathbb{N}$. Suppose that $3|n^2$ We know that 3 is prime. Then by number theory resuts we can get $3|n$ or $3|n$. We are done.

Cliam 2: $\sqrt{3}$ is irrartional.
We use inderect proof. Assume contary, $\sqrt{3}$ is rational. In other words,

$$\sqrt{3} = \frac{p}{q} \text{ for some } q \in \mathbb{Z}, q \neq 0 \text{ and } p, q \text{ have no comman factors.}$$

Thus,

$$3q^2 = p^2$$

Then, by above claim we can get that $3|p$. Thus, there exists $k \in \mathbb{Z}$ such that $3k = p$. Thus,

$$\begin{aligned} 3q^2 &= p^2 \\ 3q^2 &= (3k)^2 \\ 3q^2 &= 9k^2 \\ q^2 &= 3k^2 \end{aligned}$$

So, both p and q have comman factor 3.This contract our assumption. Thefore, $\sqrt{(3)}$ is irrational. So we are done proof of claim 2.

Since, $\sqrt{3}$ is not a rational number. Thus $\sqrt{3}$ is irrational number.

Therfore exercise 1.1 we get $\sqrt{(12)}$ is irrational. Hence, there is no rational number whose square is 12. $\square$

1.15 Proposition *The axioms for multiplication imply the following statements.*

(*a*) *If* $x \neq 0$ *and* $xy = xz$ *then* $y = z$.
(*b*) *If* $x \neq 0$ *and* $xy = x$ *then* $y = 1$.
(*c*) *If* $x \neq 0$ *and* $xy = 1$ *then* $y = 1/x$.
(*d*) *If* $x \neq 0$ *then* $1/(1/x) = x$.

The proof is so similar to that of Proposition 1.14 that we omit it.

Let $\mathbb{F}$ be an feild and let $x, y, z \in \mathbb{F}$.

a) Suppose that $x \neq 0$ and $xy = xz$

Since $x \neq 0$, there exist $(1/x) \in \mathbb{F}$ such that $x(1/x) = 1$ $(*)$

Now,

$$(xy) = (xz)$$

$$\frac{1}{x}(xy) = \frac{1}{x}(xz) \quad \text{(Property of equal sign)}$$

$$\left(\frac{1}{x}x\right)y = \left(\frac{1}{x}x\right)z \quad \text{(By assciativity property in book M3)}$$

$$1y = 1z \quad \text{(By } (*)\text{)}$$

$$y = z \quad \text{(By multiplication identity in book M4.)}$$

~~The~~

b), c), d) can be easily prooved using the result in part a)

b) We apply a) with $z=1$, then b) is trivial

c) We apply ~~b)~~ a) with $z = 1/x$.

$(x \neq 0$ and $xy = xz) \Rightarrow y = z$ [part (a)]

If $z = 1/x$

$((x \neq 0)$ and $xy = x(1/x) = 1) \Rightarrow y = 1/x$

4. Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \le \beta$.

Let $x \in E$. (Since $E \neq \phi$, there exist such x.)

Need toshow $\alpha \leq \beta$.

Since α is anupper bound aand β is an lowerbound of E. Then by definition of upper bound and lower bound (Def$^{\underline{n}}$ 1.7 in book)

$$x \leq \alpha \qquad \text{and} \qquad \beta \leq x$$

Then, $(x = \alpha \text{ or } x < \alpha)$ and $(\beta = x \text{ or } \beta < x)$

Now, we have to consider 4-Cases

Case-I If $x < \alpha$ and $\beta < x$. The by part ii) of Def$^{\underline{n}}$ 1.5

$$\beta < \alpha$$

Case-II If $x = \alpha$ and $\beta < x$ Then $\beta < \alpha = x$

Case-III If $x < \alpha$ and $x = \beta$ then

$$x = \beta < \alpha$$

Case-IV: If $x = \alpha$ and $\beta = x$ Then $x = \alpha = \beta$

Therefore $\beta < \alpha$ or $\alpha = \beta$. Thus, $\beta \leq \alpha$.

5. Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A).$$

Let $\phi \neq A \subseteq \mathbb{R}$ be a bounded below set. Let

$$-A := \{-x \mid x \in A\}$$

First of all we need to show the existance of $\sup(-A)$. ~~Since $A \neq \phi$, there exist $a \in A$.~~ Since A is bdd below, there exist $l \in \mathbb{R}$ such that,

$$a \geq l \text{ for all } a \in A.$$

Then $\quad -a \leq -l$ for all $a \in A$

$\quad -a \leq -l$ for all $-a \in -A$.

Thus, $-l$ is an upper bound of $(-A)$. Thus $\sup(-A)$ exists. (Since $\mathbb{R}$ have least upper bound property)

Now we are going to proove that $\inf(A) = -\sup(-A)$

<u>claim I</u>: $\inf(A) \geq -\sup(-A)$.

Since $\sup(-A)$ is an upper bound for $-A$

$$-a \leq \sup(-A) \quad \forall a \in A.$$

$$a \geq -\sup(-A) \quad \forall a \in A.$$

Then $-\sup(-A)$ is an lower bound for (A)

By $\text{def}^{\underline{n}}$ of infimum, $\inf(A) \geq -\sup(-A)$ ——— $\circledast$

Claim 2: $\inf(A) \leq -\sup(-A)$

Since, $\inf(A)$ is a lower bound of A.

$$\inf(A) \leq a \quad \forall a \in A.$$

$$-a \leq -\inf(A) \quad \forall a \in A.$$

$$-a \leq -\inf(A) \quad \forall -a \in -A.$$

Then $-\inf(A)$ is an upperbound for $-A$.

By def^{n} of supremum $\sup(-A) \leq -\inf(A)$. Thus

$$\inf(A) \leq -\sup(-A) \quad \text{———} \quad (**)$$

By claim 1 and 2 we get $\inf(A) = \sup(-A)$

6. Fix $b > 1$.

(*a*) If m, n, p, q are integers, $n > 0$, $q > 0$, and $r = m/n = p/q$, prove that

$$(b^m)^{1/n} = (b^p)^{1/q}.$$

Hence it makes sense to define $b^r = (b^m)^{1/n}$.

(*b*) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.

(*c*) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leq x$. Prove that

$$b^r = \sup B(r)$$

when r is rational. Hence it makes sense to define

$$b^x = \sup B(x)$$

for every real x.

(*d*) Prove that $b^{x+y} = b^x b^y$ for all real x and y.

First fix $b > 1$

a) Suppose that $m, n, p, q \in \mathbb{Z}$ with $n > 0, q > 0$ and $r = \frac{m}{n} = \frac{p}{q}$

We are going to use thm 1.21 uniquenes property

Recall thm 1.21

$\forall x \in \mathbb{R}^+ \; \forall n \in \mathbb{Z}^+ \; \exists!\, y \in \mathbb{R}$ such that $y^n = x$.

Now consider,

$$\cancel{\left(\left(b^m\right)^{1/n}\right)^{nq} = \left(\left(\left(b^m\right)^{1/n}\right)^n\right)^q = \left(b^m\right)^q = b^{mq}}.$$

~~Let~~ First note that $\frac{m}{n} = \frac{p}{q}$ implies $np = mq$.

$$\left(\left(b^m\right)^{1/n}\right)^{nq} = \left(\left(\left(b^m\right)^{1/n}\right)^n\right)^q = \left(b^m\right)^q = b^{mq} = b^{np}$$
$$= b^{pn} = \left(b^p\right)^n = \left(\left(\left(b^p\right)^{1/q}\right)^q\right)^n$$
$$= \left(\left(b^p\right)^{1/q}\right)^{nq}$$

Now by theorm 1.21, we get

$$\left(b^m\right)^{1/n} = \left(b^p\right)^{1/q} \qquad \blacksquare$$

Hence it makes sense to define $b^r = \left(b^m\right)^{1/n}$

Suppose that $r, s \in \mathbb{Q}$. Then $r = \frac{m}{n}$ and $s = \frac{p}{q}$ forsome $m, n, p, q \in \mathbb{Z}$ with $n, q \neq 0$.

Now consider

$$\left(b^{r+s}\right)^{nq} = \left(b^{\frac{m}{n}+\frac{p}{q}}\right)^{nq} = \left(b^{\frac{mq+np}{nq}}\right)^{nq}$$

$$= \left(\left(b^{mq+np}\right)^{1/nq}\right)^{nq}$$

$$= b^{mq+np}$$

$$= b^{mq} \cdot b^{np}$$

$$= \left(b^{m}\right)^{q} \cdot \left(b^{p}\right)^{n}$$

$$= \left(\left(\left(b^{m}\right)^{1/n}\right)^{n}\right)^{q} \cdot \left(\left(\left(b^{p}\right)^{1/q}\right)^{q}\right)^{n}$$

$$= \left(\left(b^{m}\right)^{1/n}\right)^{nq} \cdot \left(\left(b^{p}\right)^{1/q}\right)^{nq}$$

$$= \left(\left(b^{m}\right)^{1/n} \cdot \left(b^{p}\right)^{1/q}\right)^{nq}$$

$$= \left(b^{r} \cdot b^{s}\right)^{nq}$$

By thm 1.21, weget $b^{r+s} = b^{r} \cdot b^{s}$

c/d - Redo It

7. Fix $b > 1$, $y > 0$, and prove that there is a unique real x such that $b^x = y$, by completing the following outline. (This x is called the *logarithm of y to the base b*.)

(*a*) For any positive integer n, $b^n - 1 \geq n(b - 1)$.

(*b*) Hence $b - 1 \geq n(b^{1/n} - 1)$.

(*c*) If $t > 1$ and $n > (b - 1)/(t - 1)$, then $b^{1/n} < t$.

(*d*) If w is such that $b^w < y$, then $b^{w+(1/n)} < y$ for sufficiently large n; to see this, apply part (*c*) with $t = y \cdot b^{-w}$.

(*e*) If $b^w > y$, then $b^{w-(1/n)} > y$ for sufficiently large n.

(*f*) Let A be the set of all w such that $b^w < y$, and show that $x = \sup A$ satisfies $b^x = y$.

(*g*) Prove that this x is unique.

Fix $b>1, y>0$

a) claim 1: $\forall n \in \mathbb{Z}^+, b^n - 1 \geq n(b-1)$

We are going to use mathematical induction.

~~$n=1$ $b^1 - 1 \geq 1(b-1)$~~

If $n=1$, the inequlity holds

Assume that the given inequlity holds if $n = p \in \mathbb{Z}^+$, i.e:

$$b^p - 1 = p(b-1)$$

$\boxed{\text{Now } n = p+1}$

$$\begin{aligned} b^{p+1} - 1 &= b^{p+1} - b + (b-1) \\ &= b(b^p - 1) + (b-1) \\ &\geq b(p(b-1)) + (b-1) \quad \text{(By induction hypothesis)} \\ &= bp(b-1) + (b-1) \\ &= (bp+1)(b-1) \\ &\geq (p+1)(b-1) \quad (\text{Since } b>1) \end{aligned}$$

Therefore, by mathematical induction, forall $n \in \mathbb{Z}^+$ such that $b^n - 1 \geq n(b-1)$,

b) Now replace b with $b^{1/n}$. Then ~~g1 fist giv~~ part a) inequlity becomes.

$$(b-1) \geq n\,(b^{1/n}-1).$$

Note that $b^{1/n} > 1$

c) Now assume that $t > 1$ and $n > \frac{(b-1)}{(t-1)}$

Thus, $\cancel{t}\, n(t-1) > (b-1)$

The part b) becomes,

$$n(t-1) > (b-1) \geq n\,(b^{1/n}-1)$$

$$n(t-1) > n(b^{1/n}-1)$$

$$(t-1) > (b^{1/n}-1) \quad , (\because n \in \mathbb{Z}^{+})$$

$$t > b^{1/n}$$

d) Now suppose that $b^w < y$. Then

$$1 < y b^{-w}$$

Apply part(c) with $t = y b^{-w} > 1$

and assume that n is large with

$$n > \frac{(b-1)}{(t-1)},$$

$$b^{1/n} < y b^{-w}$$

$$b^{w+1/n} < y \quad (\because b^{w} > 0)$$

e) Now Suppose that $b^w > y$. Then $b^w\left(\frac{1}{y}\right) > 1$

Apply part c) $t = b^w/y$

~~assu~~ Furthure, assume that $n > \frac{b-1}{t-1}$

By result of part c), $b^{1/n} < b^w\left(\frac{1}{y}\right)$

$$y b^{1/n} < b^w$$

$$y < b^{w-1/n}$$

f) Let $A := \{ w \mid b^w < y \}$

Let x = sup(A). Since $\mathbb{R}$ is ordered set, one of ~~them~~ is following is true

$$b^x < y \quad \cancel{\text{or}}, \quad b^x = y, \quad b^x > y$$

Case-1 If $b^x < y$, in part d, we showed that for sufficient large n

$b^{x+\frac{1}{n}} < y$. Then $x+\frac{1}{n}$ ~~is an upper bdd~~ $\in A$.

This contradic the fact x is the sup(A)

<u>Case-II:</u> If $b^x > y$, in part e:, we already showed that for sufficiently large n, $b^{x-1/n} > y$

Thus $\left(x-\frac{1}{n}\right)$ ~~is an lower bound~~ is an upper bound for A. This contradict x is $\sup(A)$. (least upper bound of A)

Therefore, $b^x = y$, $x = \sup(A)$

9) Redo It